LECTURES ON SHIMURA CURVES 9: QUATERNION ORDERS

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1. Orders and ideals in quaternion algebras

Our task here is to recall part of the theory of orders and ideals in quaternion algebras. Some of the theory makes sense in the context of B/K a quaternion algebra over a field K which is the quotient field of a Dedekind ring R. For our purposes K will always be a number field, or the completion of a number field at a finite prime, and R will be the ring of integers of K. (Nevertheless, we shall see that in the global case, the most important distinguishing feature of B is its non/splitting at the infinite places.)

Definition: An element $x \in B$ is said to be integral (with respect to R) if its (reduced) characteristic polynomial $T^2 - t(x)T + n(x)$ has R-coefficients. This is consistent with the notion of integrality from *commutative* algebra: i.e., it would be equivalent to require $R[x]$ to be a finitely generated R-module. Indeed, $R[x]$ is a commutative R-algebra, so we have not yet left the realm of commutative algebra.

Rather, what differs from the commutative case is that the set of integral elements of H need not form a ring.

Example 1: Take $B = M_2(\mathbb{Q})$ and consider the matrices

$$X = \begin{bmatrix} \frac{1}{2} & 3 \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix}, \; Y = \begin{bmatrix} 0 & \frac{1}{5} \\ 5 & 0 \end{bmatrix}.$$

Then X and Y are integral elements but neither $X + Y$ nor XY are integral.[1] As we shall soon see, this makes the theory of quaternion orders signficantly more complicated than in the commutative case.

First we give a long list of definitions.

Definition: An *ideal* of H is just an R-submodule I of B such that the natural map $I \otimes_R K \to B$ is an isomorphism. (In other words, it is the analogue of a fractional ideal in the commutative case.) An ideal is said to be integral if it consists of integral elements. An *order* $\mathcal{O}$ is an ideal which is a subring.

Exercise 1: Show that an order is an integral ideal.

An *maximal order* is (of course) an order which is not properly contained in any other order, and an *Eichler order* is an order which can be written as the interseection of two maximal orders.

[1] I "found" this example in my thesis, but it must come originally from Vigneras' book.

Note that since the set of all integral elements is (in general) not an order of B, something must be said about why orders exist at all.

Lemma 1. *Any integral element x of H lies in an order, hence in some maximal order.*

Proof: If $x \in R \subset K \cdot 1$, then it will lie in all orders, so it suffices to handle the case where x is not in R. In this case, $K(x) = L$ is a quadratic extension in which $\mathfrak{o} = R[x]$ is an R-order.[2] Thus we can write $x = a + \sqrt{c}$ for $a, c \in R$, and it is clearly enough to construct a maximal order containing $x := \sqrt{c}$. Let y be an R-integral Noether-Skolem element: i.e., such that $y^{-1}xy = \overline{x}$. (Note that the choice of such a y is unique up to a scalar from K, and we can take it to be integral just by clearing the denominator.) As we have already seen, the R-submodule generated by x and y is $R + Rx + Ry + Rxy$: in particular, it is an R-order. There is no reason for it to be maximal, but by Zorn's Lemma every order is contained in a maximal order. (Shortly we will give a better proof of this last step.)

Another way of proving the lemma would exploit the relationship between ideals and orders. Namely, if $I \subset B$ is an ideal, we define its left and right orders

$$\mathcal{O}_\ell(I) = \{h \in B \mid hI \subset I\},$$
$$\mathcal{O}_r(I) = \{h \in B \mid Ih \subset I\}.$$

Note that these are orders, not necessarily equal. An ideal is *two-sided* if $\mathcal{O}_\ell(I) = \mathcal{O}_r(I)$. Exercise 2: For an ideal I, the following are equivalent:
a) I is contained in $\mathcal{O}_\ell(I)$ and $\mathcal{O}_r(I)$.
b) $II \subset I$.
c) I is an integral ideal.

An ideal I is *principal* if there exists an $h \in B$ such that $I = \mathcal{O}_\ell(I)b = b\mathcal{O}_d(I)$.

If I and J are two-sided ideals, the product IJ is defined in the usual way as finite sums ij. This is an ideal.

Moreover, for a two-sided ideal I we can define

$$I^{-1} := \{h \in B \mid IhI \subset I\}.$$

It is easy to see that we have the containments

$$II^{-1} \subset \mathcal{O}_\ell(I), I^{-1}I \subset \mathcal{O}_r(I).$$

We would very much like these inclusions to be equalities: if so, we would feel justified in calling I^{-1} the inverse of I. Note that a principal ideal is invertible. Later we will introduce the notino of a *locally principal* ideal, which is also easily seen to be invertible. For our purposes here (ideals of Eichler orders of local and global fields), all ideals will be locally principal, hence invertible. Ideal classes: we say two ideals I, J are equivalent on the left if $I = hJ$ for some $h \in B$. If $\mathcal{O}$ is an order, we define the set $\mathrm{Pic}_\ell(\mathcal{O})$ of left-ideal classes of $\mathcal{O}$: this is the set of ideals with *right* order $\mathcal{O}$ modulo equivalence on the left. (This is the correct way to do

[2] We will use upper case caligraphic letters to denote orders of H and lowercase gothic lettters to denote orders of quadratic subalgebras of H.

it, since modifying an ideal on the left does not change its right order.) Of course there is a similar definition $\mathrm{Pic}_r(\mathcal{O})$ f of right-classes of left $\mathcal{O}$-ideals. This does not give anything new, since the map $I \mapsto I^{-1}$ induces a bijection $\mathrm{Pic}_\ell(\mathcal{O}) \to \mathrm{Pic}_r(\mathcal{O})$.

Two orders are said to be of the same *type* if they are conjugate by an element $b \in B^\times$.

Linked orders: We say that two orders $\mathcal{O}$ and $\mathcal{O}'$ are *linked* if there exists an ideal I whose left order is $\mathcal{O}$ and whose right order is $\mathcal{O}'$. This is an equivalence relation, and we will speak of *linkage classes* of orders. As an example, the maximal orders lie in a single linkage class (since, if $\mathcal{O}$ and $\mathcal{O}'$ are any two orders, put $I := \mathcal{O} \cdot \mathcal{O}'$ and $\mathcal{O} \subset \mathcal{O}_\ell(I)$ and $\mathcal{O}' \subset \mathcal{O}_r(I)$; if $\mathcal{O}$ and $\mathcal{O}'$ are maximal, we must have equality).

Lemma 2. *Linked orders have the same number of (left or right) ideal classes.*

Proof: Suppose $\mathcal{O}$ and $\mathcal{O}'$ are linked by I. We define a map from the set of left $\mathcal{O}$-ideals to the set of right $\mathcal{O}'$-ideals by $J \mapsto J^{-1}I$. The map $P \mapsto IP^{-1}$ gives an inverse. Moreover, the map descends to ideal classes, since $Jh \mapsto (Jh)^{-1}I = h^{-1}J^{-1}I$.

Definition: The *class number* of B (with respect to R) is $\#\,\mathrm{Pic}_\ell(\mathcal{O})$, where $\mathcal{O}$ is any maximal order. Note that the preceding lemma ensures that this is well-defined. The *type number* of B is the number of conjugacy classes of maximal orders of B.

Lemma 3. *The following are equivalent:*
a) Two orders $\mathcal{O}$ and $\mathcal{O}'$ are of the same type.
b) There exists a principal ideal I linking $\mathcal{O}$ and $\mathcal{O}'$.

Corollary 4. *If the class number of B is one, its type number is one.*

Proof: Since any two maximal orders are linked by some ideal, this follows immediately from the lemma.

Exercise X: Show that, in general, the type number is less than or equal to the class number.

Our goal here is to compute class numbers and type numbers for quaternion algebras over p-adic fields and over number fields.

1.1. **Discriminants.** One aspect of the theory of orders and ideals which works just as nicely in the commutative case (thank goodness) is the discriminant. Let I be an ideal of B. We define $n(I)$ to be the fractional R-ideal generated by reduced norms of elements of I. For an order $\mathcal{O}$, we define as usual the *different* $D(\mathcal{O})$, as the fractional ideal which is the inverse of the dual $\mathcal{O}^\star$ of $\mathcal{O}$ for the trace form:

$$\mathcal{O}^\star := \{x \in B \mid t(x\mathcal{O}) \subset R\}.$$

We can define the discriminant $\Delta(\mathcal{O})$ as the norm of the different ideal.

Discriminants can be computed using the following two useful facts:

(i) if $\mathcal{O}$ is a free R-module with basis v_i, then $\Delta(\mathcal{O})^2$ is the principal ideal $R(\det(t(v_i v_j))$; (ii) Δ can be computed locally on R. Remark: Note that we computed the discriminant using the *reduced norm* and the *reduced trace*. This is really a reduced discriminant in the following sense: the associated (norm) form on $\mathcal{O}$ is a quaternary quadratic form over the ring R. The discriminant of this quadratic form in the usual sense (i.e., the determinant of a matrix representation) is easily seen to be a square, and our discriminant is its square root. (So in particular, the squareclass of the discriminant of an order *does* depend on the order.)

Proposition 5. *Let $\mathcal{O} \subset \mathcal{O}'$ be orders of B. Then $\Delta(\mathcal{O}') \subset \Delta(\mathcal{O})$, with equality iff $\mathcal{O}' = \mathcal{O}$.*

Exercise 2: Prove it.

Note that this gives a much more reasonable explanation for why every order is contained in a maximal order: R is a Noetherian ring!

Example 2: Take $B = M_2(K)$ and $\mathcal{O} = M_2(R)$. Taking the obvious standard basis of $\mathcal{O}$, we compute immediately that the discriminant ideal of $\mathcal{O}$ is R itself, which implies that $M_2(R)$ is a maximal order. (Stop and convince yourself that it is *never* the unique maximal order.)

Example 3: Take $K = \mathbb{Q}$, $B = \langle -1, \ -1 \rangle$. As we've just recalled, the Hilbert symbol representation determines an integral basis $\mathcal{O} = \mathbb{Z}[1, i, j, ij]$, and one calculates that the dsicriminant of $\mathcal{O}$ is $4\mathbb{Z}$, which is of course not a maximal ideal of $\mathbb{Z}$. Indeed, a maximal order containing it is $\mathcal{O}' = \mathbb{Z}[1, i, j, \frac{1+i+j+ij}{2}]$ (here one must check that this is actually an order, although this is certainly a well known classical fact), with discriminant $2\mathbb{Z}$. This is a maximal order, and it is no accident that its discriminant agrees with the discriminant of B in the sense of Brauer groups (i.e., the product of the ramified finite primes).

The determination of class and type numbers of maximal orders in a quaternion algebra over a number field K proceeds by a familiar two-step process. Step 1 is to understand what happens at every completion of K. Step 2 is to figure out how the local results fit together to give a global result, namely what (if any) obstructions intervene. We will discuss the two steps in the next two sections.

2. Local case

Let $K = K_v$ be a p-adic field, and let B/K be a quaternion algebra. There are only two cases: either $B \cong M_2(K)$, or it is the unique division quaternion algebra. We will recall some results in each case. We will for the most part not give proofs. Such proofs can be found in Vignéras' book. On the other hand, many of these results are "so reasonable" that providing the proofs would probably be good exercises for the reader.

Recall $R = \mathcal{O}_K$. Write π for a uniformizing element of K.

2.1. **Split case.**

Let V/K be a two-dimensional vector space, so we are interested in studying ideals and orders in $B = \mathrm{End}(V)$.

Proposition 6. *a) The maximal orders of* $\mathrm{End}(V)$ *are the rings* $\mathrm{End}(L)$*, where* L *is a complete* R*-lattice of* V*.*
b) The ideals of these maximal orders are all of the form $\mathrm{Hom}(L, M)$*, where* L *and* M *are complete* R*-lattices of* V*.*

Theorem 7. *a) All maximal orders of* $M_2(K)$ *are conjugate to* $M_2(R)$*.*
b) The two-sided ideals of $M_2(R)$ *form a cyclic group generated by the prime ideal* $P = M_2(R)\pi = \pi M_2(R)$*.*
c) The integral left $M_2(R)$*-ideals are the distinct ideals* $M_2(R)\begin{bmatrix} \pi^n & r \\ 0 & \pi^m \end{bmatrix}$*, where* n *and* m *are non-negative integers and* r *runs through a set of coset representatives of* $R/\pi^m R$ *in* R*.*

Let $\mathcal{O} = \mathrm{End}(L)$, $\mathcal{O}' = \mathrm{End}(M)$ be two maximal orders of V)$.If$x, y $\in K^\times$, note that $\mathrm{End}(Lx) = \emptyset$, $\mathrm{End}(My) = \mathcal{O}'$, so that the maximal order depends upon the lattice only up to homothety.

Exercise: a) Show that, conversely, if $\mathrm{End}(L) = \mathrm{End}(M)$ for two lattices in V, then L and M are homothetic. (Suggestion: the discussion of the following paragraph is helpful here.)
b) Conclude that the set of maximal orders in a split quaternion algebra over a p-adic field K is in bijection to the homogeneous space $GL_2(K)/Stab(M_2(\mathcal{O}_K)) = GL_2(K)/K^\times GL_2(\mathcal{O}_K) = PGL_2(K)/PGL_2(\mathcal{O}_K)$.

If L and M are any two lattices, the theory of elementary divisors gives us a basis (f_1, f_2) of L such that for some integers a, b, $(\pi^a f_1, \pi^b f_2)$ is a basis for M. The integer $|b-a|$ is independent of the scaling, so we may define the *distance* between two maximal orders $\mathcal{O}, \mathcal{O}'$ to be this quantity $|b-a|$. As an example, the distance between $M_2(R)$ and $\begin{bmatrix} R & \pi^n R \\ \pi^n R & R \end{bmatrix}$ is n. We define an Eichler order of local level n to be an order obtained by intersecting two maximal orders of distance n.

Lemma 8. *Let* $\mathcal{O}$ *be an order of* $M_2(K)$*. The following are equivalent:*
a) There exists a unique pair of maximal orders $\mathcal{O}_1$*,* $\mathcal{O}_2$ *such that* $\mathcal{O} = \mathcal{O}_1 \cap \mathcal{O}_2$*.*
b) $\mathcal{O}$ *is an Eichler order.*
c) there exists a unique non-negative integer n *such that* $\mathcal{O}$ *is conjugate to*

$$\mathcal{O}_n := \begin{bmatrix} R & R \\ \pi^n R & R \end{bmatrix}.$$

Exercise: a) Consider the graph whose vertices are the maximal orders of $M_2(K)$ and such that two vertices are connected by a (single) edge if and only if the two maximal orders have distance one. Show that this graph is the homogeneous tree of order $q+1$ (where q is the cardinality of the residue field of K), called the Bruhat-Tits tree of $PGL_2(K)$.

Exercise: Show that the discriminant of a level n Eichler order is (π^n). (Here we may view a maximal order as an Eichler order of level 0.)

Let us summarize: for the split quaternion algebra over a p-adic field, there exist infinitely many maximal orders but they are all conjugate (i.e., the type number is 1); the class number is equal to 1 (i.e., every left ideal for a maximal order is principal); and the only invariant necessary to classify conjugacy classes of Eichler orders is the level, which can be any positive integer.

Remark: The constructions of this section can be generalized in any number of ways, i.e., to $GL_n(K)$ instead of $GL_2(K)$. Direct further questions to Gil Alon.

2.2. **Nonsplit case.** Now let B be the division quaternion algebra over a p-adic field K. Things actually work out even more nicely in this case:

Proposition 9. *The set of integral elements of B forms an order, which is necessarily the unique maximal order of B.*

Here the proof is rather interesting, and we will sketch it. Essentially, the idea is to extend the discrete valuation v on K to a valuation on B. In fact we do this in exactly the same way that one extends a valuation to a commutative extension: namely, we define the map $v : B^\times \to \mathbb{Z}$ by $v(x) := v(n(x))$. This gives a group homomorphism which, when restricted to K, is precisely twice the original valuation. On the other hand, $v(\sqrt{\pi}) = 1$, so the valuation is surjective onto $\mathbb{Z}$.

Exercise: How do we know there is a square root of π in B? (Hint: An extension of local fields L/K induces a restriction map $(L) \to (K)$, or a map from $\mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$. What is this map?)

It is easy to see that this deserves to be called a valuation; namely it is multiplicative-to-additive and satisfies $v(x + y) \geq \inf(v(x), v(y))$. (See Serre's *Corps Locaux* for more details.) Note that if $x \in B \setminus K$, then the valuation restricted to the quadratic extension field $L = K(x)$ is either the natural $\mathbb{Z}$-valued valuation on L (if L is ramified) or twice it (if L is unramified); in particular the set of integral elements of B can be characterized as those having non-negative valuation. But clearly the set of elements of B having non-negative valuation forms a ring; call it $\mathcal{O}_B$.

From this uniqueness, it follows that every integral ideal of $\mathcal{O}_B$ is two-sided. Indeed, there is a unique maximal ideal P, which is principal, consisting of (guess what?) the elements of positive valauation, and the complete set of ideals is $\{P_i\}_{i\geq 1}$.

Exercise: Show that the discriminant of the maximal order is (π).

Thus the class number and the type number is 1. Moreover, every Eichler order is maximal.

3. Main results for orders over global fields

Let K be a number field, B/K a quaternion algebra, and let $K_B \subset K$ be the set of elements which are non-negative at every ramified real place of B.

Theorem 10. *We have $K_B = n(B)$. In particular, if B is split at every real place of K, $n(B) = K$.*

Proof: As remarked several times in class, this is really a piece of the theory of quadratic forms. It follows from (i) the (weak) Hasse principle for quadratic forms: i.e., a qaudratic form q over a number field K represents an element a of K if and only if for every place v of K, $q \otimes K_v$ represents a; and (ii) the fact that a quadratic form of dimension at least 4 over a p-adic field is universal (i.e., reprsents everything).

Theorem 11. *Let H be the algebraic group of norm* 1 *elements of B (sometimes called $SL_1(D)$ in the notes). Let S be a set of places of K containing at least one Archimedean place. Write $H_S = \prod_{v \in S} H(K_v)$. Then, if H_S is* not *compact, $H(K)H_S$ is dense in $H(A)$.*

Proof: This is the strong approximation theorem applied to the group H (and not for the first time!).

In practice we will want to apply this theorem with S being precisely the set of Archimedean places, and in that case the noncompactness hypothesis is equivalent to B being split at some infinite place (because $SL_2(\mathbb{R})$ is compact, whereas the group of norm 1 elements in the Hamiltonian quaternions is homeomorphic to the three-sphere). This gives us a fundamental dichotomy for quaternion algebras:

Definition: Say that a quaternion algebra B/K is *not totally definite* (or ntd) if there exists a split Archimedean place; otherwise it is totally definite (td).[3] Note that if K is not totally real every qa is ntd (but in practice we will be interested only in the totally real case, and especially in $K = \mathbb{Q}$).

Localization: let I be an ideal in B. For each finite place v of K, we can associate the ideal $I_v = I \otimes_R R_v$ of $B_v = B \otimes K_v$. Note that for all but finitely many places v, I_v is an integral ideal of determinant 1, hence is a maximal order of $B_v \cong M_2(K_v)$.

One has the notion of a local property of ideals (resp. orders), namely a property of I (resp. $\mathcal{O}$) which can be checked on the localizations I_v.

Exercise: Show that the following properties of ideals and orders are local: being an order, being a maximal order, being an Eichler order, being an integral ideal, being a two-sided ideal, the discriminant.

The fact that the discriminant can be computed locally leads to the following important facts:

Proposition 12. *Let $\mathcal{O}$ be an Eichler order in a quaternion algebra over a number field K. Then $\Delta(\mathcal{O}) = \mathcal{N} \cdot \mathcal{D}$, where $\mathcal{D}$ is the squarefree product of the ramified primes of B and $\mathcal{N}$ is coprime to B but otherwise arbitrary.*

In particular, an order is maximal if and only if its discriminant is equal to the Brauer group discriminant.

[3] The terminology n/td is *ad hoc*, but seems btter than Vignéras terminology: "B verifies (does not verify) Eichler's condition."

Adelization: Fix an ideal J of B of discriminant δ. For v not dividing δ, J_v is a maximal order of the split quaternion algebra $M_2(K_v)$. We will call it the "standard lattice."

Exercise 4: The map $I \mapsto (I_v)$ is injective, with image equal to systems of local lattices which are *standard* at all but finitely many v's.

Now let I be an ideal of B, and let $g \in B^\times(A)$; in other words, $g = (g_v)$ is a collection of local elements $g_v \in B_v^\times$ with the property that except for a finite set of v, $g_v \in GL_2(R_v)$. Consider the system $(g_v I_v)$ of local lattices; except for a finite set of v's, $g_v I_v = I_v$. Thus by the preceding exercise, the system $\{g_v I_v$ determines another ideal, and all such ideals arise in this way.

Remark: We could have worked with $B^\times(A_f)$ instead of $B^\times(A)$ and gotten the same result. However, for certain constructions to come, we will want the larger group. From this we can deduce a **global - adelic dictionary**. Fix $\mathcal{O}$ a level N Eichler order of B, and put

$$\mathcal{O}_A^\times := (\mathcal{O} \otimes \hat{\mathbb{Z}})^\times.$$

Then the space $\mathcal{O}_A^\times \backslash B^\times(A)$ classifies left $\mathcal{O}$-ideals. Indeed, take $J = \mathcal{O}$, and to every $g = (g_v) \in B_A^\times$, we take I to be the ideal associated to the system $I_v g_v$ (note that we have multiplied on the right!); this is still a left $\mathcal{O}$-ideal. A local system (g_v) stabilizes $\mathcal{O}$ if and only if it stabilizes $\mathcal{O}_v$ for all v, i.e., if and only if it is an element of $\mathcal{O}_v^\times$: done.

We write $N(\mathcal{O}_A)$ for the normalizer of $\mathcal{O}_A^\times$ in $B_A^\times$. Then we have the following equivalences:

Proposition 13. *Let $\mathcal{O}$ be a level N Eichler order in the quaternion algebra B over the number field F. Then:*
(a) Two sided $\mathcal{O}$-ideals correspond to $\mathcal{O}_A^\times \backslash N(\mathcal{O}_A)$.
(b) Level N Eichler orders correspond to $N(\mathcal{O}_A) \backslash B_A^\times$.
(c) $\operatorname{Pic}_r(\mathcal{O})$ corresponds to $\mathcal{O}_A^\times \backslash B_A^\times / B_K^\times$.
(d) Types of level N Eichler orders correspond to $N(\mathcal{O}_A) \backslash B_A^\times / B_K^\times$.

Exercise X: Prove it.

Part c) is especially interesting: it is the non-commutative analogue of the classfield theoretic computation of the Picard group of an order $\mathfrak{o}$ in a number field K:

$$\operatorname{Pic}(\mathfrak{o}) = \mathfrak{o}_A^\times \backslash K_A^\times / K^\times.$$

In fact it is more than an analogy: let $K_B^\times$ be the elements of K which are totally positive at every ramified place of B; let (K) be the group of fractional ideals of K, and let P_H be the subgroup of principal ideals with generators in $K_B^\times$.

Define $h_B := \#(K)/K_B^\times$. if h denotes the class number of K and h^+ the narrow class number, then we have $h \mid h_B \mid h^+$.

Theorem 14. *(Eichler) Let B/K be a quaternion algebra which is not totally definite, and let $\mathcal{O}$ be an Eichler order in B. Then the reduced norm map induces a bijection $n : \operatorname{Pic}_r(\mathcal{O}) \to (K)/P_H$. In particular, h_B is the class number of H.*

Proof: First note that the reduced norm map $n : B^\times \to K^\times$, when evaluated at the adelic points, induces a map $B_A^\times \to K_A^\times$. This descends to a map

$$n : \mathcal{O}_A^\times \backslash B_A^\times / B_L \to R_A^\times \backslash K_A^\times / K_B.$$

Moreover, the right hand side is the adelic version of the group $(K)/K_B^\times$ (i.e., global class field theory says the two groups are canonically isomorphic). It remains to be seen that the map is injective and surjective. The surjectivity is easy: at every place v of K except a ramified infinite place, we have $n(B_v^\times) = K_v^\times$, and at every finite place we have $n(\mathcal{O}_v^\times) = R_v^\times$. Moreover, at every infinite place v, by $\mathcal{O}_v^\times$ we mean $B_v^\times$ and by $R_v^\times$ we mean $K^\times$. So no problem.

For the injectivity, we need the Strong Approximation Theorem: recall that $B^1_{/K}$, the elements of B of reduced norm 1, is a simply connected semisimple algebraic group over K. Let S be the set of Archimedean places of K. The Strong Approximation Theorem,[4] as discussed earlier in the course, says that if $B_S^1 := \prod_{v \in S} B^1(K_v)$ is not compact, then $B^1(K)B_S^1$ is dense in B_A^1. Now, since $\mathcal{O}_A^\times$ is an open subgroup of $B_A^\times$ containing B_S^1, we necessarily have $B_A^1 \subset \mathcal{O}_A^\times B_K^\times$. This proves the injectivity and completes the proof of Eichler's theorem.

Corollary 15. *Let F be a totally real field and $B_{/F}$ a quaternion algebra over F which is ntd (not totally definite).*
a) If F has narrow class number 1, then all level $\mathcal{N}$ Eichler orders of B are conjugate, and every (left or right) ideal of an Eichler order is principal.
b) If B is totally indefinite, the same conclusion holds when F has class number one.

Remarks:
(i) One can say much more about the relationship between the type number and the class number in the ntd case: see Vignéras book. In my thesis, I recorded the following striking result recorded: right $h = \mathrm{Pic}_r(\mathcal{O})$ for the class number of the Eichler order $\mathcal{O}$, t for the type number, and h_2 for the number of classes of two-sided $\mathcal{O}$-ideals. Then $h = th_2$. One can ask for an interpretation of h_2 as the cardinality of a subgroup of $(K)/P_H$ corresponding under Eichler's theorem to classes of two-sided ideals. The subgroup in question is the one generated by: (a) squares of ideals of $R = \mathfrak{o}_K$, (b) prime ideals ramifying in H, and (c) prime ideals dividing the level $\mathcal{N}$ to an odd power. From this one gets the following

Corollary 16. *If $h_B = \#(K)/P_H$ is odd, there is a unique conjugacy class of Eichler order of any given level.*

Thus, when h_B is odd, one can speak of "the Shimura curve $X_0^\Delta(\mathcal{N})$, i.e., the quotient of $\mathcal{H}$ by the order $\Gamma(B, \mathcal{O})$ where $\mathcal{O}$ is an Eichler order of level N. (In particular, we can and do speak this way when $K = \mathbb{Q}$!)

(ii) These theorems are highly relevant to the determination of the set of connected components of

$$V(\mathcal{O}) = \mathcal{O}^\times \backslash (\mathcal{H}^\pm)^g \times B^\times(A_f)/\mathcal{O}(\hat{\mathbb{Z}}^\times),$$

[4] Actually we stated the Strong Approximation Theorem only for algebraic groups over $\mathbb{Q}$. You can either (i) believe that it holds verbatim over an arbitrary number field K, or (ii) deduce the theorem for K fro the theorem for $\mathbb{Q}$ by a Weil restriction argument: your choice.

although we will not have time to return to this issue in this course.

(iii) Although it is very tempting to make life simpler by taking $h_B = 1$, the general case contains some very interesting mathematics. Indeed, suppose that $\mathcal{O}$ and $\mathcal{O}'$ are two Eichler orders of the same level $\mathcal{N}$ but not of the same type (i.e., not conjugate by an element of $B^\times$). Then the two Shimura curves associated to $\mathcal{O}$ and $\mathcal{O}'$ are non-isomorphic (there is an argument to be made here about normalizers of arithmetic groups inside real Lie groups; the statement of the theorem will appear in the notes on arithmetic groups). We can choose B so that $\mathcal{O}$ and $\mathcal{O}'$ have no elliptic points, in which case the Riemannian metric on $\mathcal{H}$ descends to $S(\mathcal{O})$ and $S(\mathcal{O}')$: these are compact Riemannian surfaces: in particular, there is a laplace operator Δ whose eigenvalues form the *spectrum*. It has long been an important problem in Riemannian geometry to construct isospectral but non-isometric Riemannian manifolds. It turns out that the isospectrality of $S(\mathcal{O})$ and $S(\mathcal{O}')$ can be interpreted in terms of the arithmetic of the quaternion orders (having the same number of conjugacy classes with any given characteristic polynomial, or something much like that). By studying arithmetic of Eichler orders in sufficient detail, Vignéras was able to show that there exist arbitrarily large sets of Eichler orders of level $\mathcal{N}$ (for suitable $\mathcal{N}$, F and B) such that the corresponding Shimura curves are mutually isospectral but pairwise non-isometric. This was the first example of a pair of isospectral non-isometric Riemannian surfaces!

(iv) The corollary (and hence Eichler's theorem) fails dramatically for totally definite quaternion algebras: the class number of a totally definite quaternion algebra over $\mathbb{Q}$ of discriminant D goes to infinity with D. These definite quaternion algebras will arise in our story as well: they are intimately related to elliptic curves and modular forms.