COMPLEX ANALYSIS: SUPPLEMENTARY NOTES

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Contents

Provenance

These are lecture notes from my Fall 2017 Math 4150/6150 Complex Variables course. This is a one semester course at the advanced undergraduate level. The main prerequisite is some very elementary real analysis at the level of the Math 3100 (sequences and series) course that is taught at UGA. In places we also use a bit of linear algebra.

I am not myself an expert in complex analysis, so I want to make clear that in writing these notes I am almost exclusively following other sources. The main text for the course was [BMPS], and overall we follow this rather closely, in particular in the ordering of topics. Perhaps it will be helpful to list the main differences between these notes and [BMPS]:

• We introduce the complex logarithm rather late in the day, in §4.9.
• We give a proof of the equivalence of connectedness and path-connectedness for open subsets of $\mathbb{C}$, following a slick argument of [FB].
• Our discussion of contour integration is largely motivated by analogies to line integrals in multivariable calculus.
• We do not make use of the notion of homotopy of paths. (We certainly have nothing against it! But we think these ideas are better developed in either a second course on complex analysis or a first course in algebraic topology.) Our notion of **simple connectedness** is therefore in terms of simple closed curves being the boundaries of subdomains. We assume the Jordan-Schoenflies Theorem that every simple closed curve has a well-defined interior and exterior and that the interior is simply connected. Other than this, our discussion is self-contained and complete.
• We include a proof of the Cauchy-Goursat Theorem, largely following [SS].
• We include a proof of Cauchy's Integral Theorem for Derivatives before our discussion of series methods. On the other hand, we develop the basic theory of series a bit differently so that Cauchy's Integral Theorem for Derivatives also follows independently from this theory, whereas the development in [BMPS] uses Cauchy's Integral Theorem for the First Derivative.
• We prove the Maximum Modulus Theorem more directly, without using harmonic functions.
• Our discussion of computing Laurent series expansions is more detailed.
• Our discussion on summation of series using residues is more detailed. Most of the additional details are taken from [MH, §4.4].

1. The complex numbers

1.1. **Vista: The shadow of the real.** A complex number z is something of the form $z = x + iy$ where $x, y \in \mathbb{R}$ and $i^2 = -1$. We notice that this is not quite a definition in a way that was deeply problematic for hundreds of years. Not to brag, but the very first thing we will actually do is totally resolve that problem. Just for a few minutes though, let's go with it.

Complex numbers naturally arise while solving polynomial equations. The beginnings of this are taught in high school: the solution to the general quadratic equation $ax^2 + bx + c = 0$ – for a, b, c real numbers with $a \neq 0$ – is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The quantity $\Delta = b^2 - 4ac$ is called the **discriminant** of the quadratic equation. Since the image of the squaring function $\mathbb{R} \to \mathbb{R},\ x \mapsto x^2$ is $[0, \infty)$, a real number has a real square root iff it is non-negative. So if $\Delta > 0$ the quadratic equation has two distinct roots. If $\Delta = 0$ the quadratic equation has just one root, $r = \frac{-b}{2a}$, but it occurs with multiplicity 2: that is,

$$ax^2 + bx + c = a(x - r)^2.$$

However, if $\Delta < 0$ then the formula does not make sense in the real numbers. However, it does make sense in the complex numbers: $\sqrt{\Delta} = i\sqrt{|\Delta|}$. Thus complex numbers are "born to give roots to all real quadratic equations."

It turns out that the "equation solving ability" of the complex numbers is much more powerful than that. Soon enough we will see that for every nonzero complex number z and any positive integer n, there are exactly n complex numbers w such that $w^n = z$. In particular, every nonzero complex number z has exactly two complex square roots, which are negatives of each other. From this and the same quadratic formula it follows that every quadratic equation $az^2 + bz + c = 0$ with $a, b, c \in \mathbb{C}$ and $a \neq 0$ has a root in z. Later on in the course we will see that much more is true: any nonconstant polynomial with complex coefficient has a complex root: this is the celebrated Fundamental Theorem of Algebra, first proved by Gauss.

In many ways, the $\mathbb{R}$ world you have already met is just a shadow of the $\mathbb{C}$ world into which you're about to be thrust.

Example 1.1. *In the $\mathbb{R}$-world, trigonometric functions $\cos x$ and $\sin x$ are very different from exponential functions e^x: the former are periodic, hence bounded, whereas the latter grows exponentially fast. And yet there are some glimpses of a deeper relationship.*

For instance, consider one of the simplest differential equations: $y'' = \alpha y$ for $\alpha \in \mathbb{R}$. What are the solutions? If $\alpha > 0$, then they are exponential functions. If $\alpha < 0$ they are trigonometric functions.

Also consider:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!},\ \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

It looks like $\cos x$ and $\sin x$ are each "half of" e^x – but something screwy is going on with the signs.

But if we allow ourselves a **complex variable**, something amazing happens: first observe that

$$1 = i^0 = i^4 = i^8 = \ldots,$$
$$i = i^1 = i^5 = i^9 = \ldots,$$
$$-1 = i^2 = i^6 = i^{10} = \ldots,$$
$$-i = i^3 = i^7 = i^{11} = \ldots.$$

Then:

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = 1 + ix + i^2x^2/2! + i^3x^3/3! + i^4x^4/4! + \ldots$$

$$= (1 + i^2x^2/2! + i^4x^4/4! + i^6x^6/6! + \ldots) + i(x + i^2x^3/3! + i^4x^5/5! + i^6x^7/7! + \ldots)$$

$$= (1 - x^2/2! + x^4/4! - x^6/6! + \ldots) + i(x - x^3/3! + x^5/5! - x^7/7! + \ldots) = \cos x + i \sin x.$$

Thus, in the $\mathbb{C}$-world the exponential function is periodic (its period is $2\pi i$) and the trigonometric functions are unbounded. Later we will see that in the $\mathbb{C}$-world every nonconstant "entire function" is unbounded! This relationship is not only true but profoundly useful. We give one example.

Theorem 1.2. *(Fresnel[1] Integrals) We have*

$$\int_0^\infty \cos(x^2)dx = \int_0^\infty \sin(x^2)dx = \sqrt{\frac{\pi}{8}}.$$

Complex analysis will reduce this integral to one familiar from multivariable calculus:

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}.$$

1.2. **Introducing $\mathbb{C}$: something old and something new.** In this section we introduce the **complex field** $\mathbb{C}$. Recall that the real numbers $\mathbb{R}$ come endowed with binary operations of addition and multiplication

$$+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \ \cdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

that satisfy the following properties:

(P1) (Commutativity of $+$): For all $x, y \in \mathbb{R}$, we have $x + y = y + x$.
(P2) (Associativity of $+$): For all $x, y, z \in \mathbb{R}$, we have $(x + y) + z = x + (y + z)$.
(P3) (Identity for $+$): There is $0 \in \mathbb{R}$ such that for all $x \in \mathbb{R}$, we have $x + 0 = x$.
(P4) (Inverses for $+$): For all $x \in \mathbb{R}$, there is $y \in \mathbb{R}$ with $x + y = 0$.
(P5) (Commutativity of $\cdot$): For all $x, y \in \mathbb{R}$, we have $x \cdot y = y \cdot x$.
(P6) (Associativity of $\cdot$): For all $x, y, z \in \mathbb{R}$, we have $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.
(P7) (Identity for $\cdot$): There is $1 \in \mathbb{R}$ such that for all $x \in \mathbb{R}$, we have $1 \cdot x = x$.
(P8) (Inverses for $\cdot$): For all $x \in \mathbb{R}$, if $x \neq 0$, then there is $y \in \mathbb{R}$ with $x \cdot y = 1$.
(P9) (Distributivity of $\cdot$ over $+$): For all $x, y, z \in \mathbb{R}$, we have $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$.
(P10) We have $1 \neq 0$.

The complex numbers $\mathbb{C}$ will also be endowed with two binary operations of addition and multiplication and will satisfy the same field axioms. As a set, we take

$$\mathbb{C} := \mathbb{R}^2 = \{ \text{ ordered pairs } (x, y) \text{ with } x, y \in \mathbb{R}\}.$$

We tend to denote complex numbers by letters z and w, rather than using vector notation.

The addition operation on complex numbers is **something old**: it's just the usual coordinatewise addition of vectors in the plane:

$$\forall z_1 = (x_1, y_1), \ z_2 = (x_2, y_2) \in \mathbb{C}, \ z_1 + z_2 := (x_1 + x_2, y_1 + y_2).$$

In particular, we have the complex number $0 := (0, 0)$, which serves as the additive identity. As you well know, $\mathbb{R}^2$ together with the coordinatewise addition operation satisfies properties (P1)

[1] Augustin-Jean Fresnel: 1788-1827, a French engineer and physicist who pioneered the theory of optics.

through (P4). ($\mathbb{R}^2$ is a vector space over $\mathbb{R}$, and these properties hold for any vector space over $\mathbb{R}$.)

However the multiplication operation is **something new**:

$$\forall z_1 = (x_1, y_1),\ z_2 = (x_2, y_2) \in \mathbb{C},\ z_1 \cdot z_2 := (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2). \tag{1}$$

Quick question: **why**? One good answer is that it turns out (as we will soon see) that this operation satisfies properties (P5) through (P10) and thus makes $\mathbb{C}$ into a field. This is actually rather exciting, as we will now try to indicate.

Indeed, it is quite rare to be able to find a multiplication operation on vectors in $\mathbb{R}^n$ that makes it into a field. It turns out that this is only possible for $n = 1$ (i.e., the usual real numbers) and $n = 2$) (the case we're in now). There is also an interesting "near-miss": there is a multiplication operation on $\mathbb{R}^4$ that gives it all properties except for (P5), commutativity of multiplication. In order to property appreciate the "magic" of our multiplication in $\mathbb{R}^2$, we should explain why other "product" operations on vectors you've seen *do not* fit the bill.

First, for any $n \geq 1$ we have the **scalar product** of vectors: if $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$, then

$$\mathbf{x} \cdot \mathbf{y} = x_1y_1 + \ldots + x_ny_n.$$

But – as the name makes clear – this operation takes as input two vectors in $\mathbb{R}^n$ but gives as output a scalar – i.e., a real number. Thus the inner product is not a *binary operation on* $\mathbb{R}^n$. So it is not even the kind of beast we're looking for.

Second, in $\mathbb{R}^3$ we do have a binary "multiplication" operation on vectors, the **cross product**

$$(a, b, c) \times (x, y, z) := (bz - cy, -(az - cx), ay - bx).$$

However, $\mathbb{R}^3$ endowed with coordinatewise addition and the scalar product is far from a field.

Exercise 1.1. *a) Show that $\mathbb{R}^3$ endowed with coordinatewise addition and the cross product is* not *a field. (Suggestion: show that multiplication is not commutative.)*
b) Determine which of the axioms (P5) through (P10) are satisfied by $\mathbb{R}^3$ with coordinatewise addition and the cross product.

Now we return to $\mathbb{R}^2$ with the multiplication defined by (1). In order to cut down on the tedium of checking (P5) through (P10) – and also to gain further insight into the situation – we will give an alternate representation of $\mathbb{C}$. Namely, to the complex number $z = (x, y)$ we assign the 2×2 real matrix

$$M_z = \begin{bmatrix} x & -y \\ y & x \end{bmatrix}.$$

Exercise 1.2. *Let* $\mathcal{C} := \left\{ \begin{bmatrix} x & -y \\ y & x \end{bmatrix} \middle| \ x, y \in \mathbb{R} \right\}$.
a) Show: $\mathcal{C}$ is a subspace of the vector space of 2×2 real matrices.
b) Show:

$$e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ e_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

is a basis for $\mathcal{C}$.
c) Show that the map $\mathbb{C} \to \mathcal{C}$ given by $z \mapsto M_z$ is an isomorphism of vector spaces.

Part c) of the preceding exercise is hardly surprising: every two-dimensional vector space is isomorphic to $\mathbb{R}^2$. The following observation is more exciting: if $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$, then

$$M_{z_1} \cdot M_{z_2} = \begin{bmatrix} x_1 & -y_1 \\ y_1 & x_1 \end{bmatrix} \begin{bmatrix} x_2 & -y_2 \\ y_2 & x_2 \end{bmatrix} = \begin{bmatrix} x_1x_2 - y_1y_2 & -(x_1y_2 + x_2y_1) \\ x_1y_2 + x_2y_1 & x_1x_2 - y_1y_2 \end{bmatrix} = M_{z_1 \cdot z_2}.$$

In other words, under the isomorphism from $\mathbb{C}$ to $\mathcal{C}$ given by $z \mapsto M_z$, the "new" product operation we've defined corresponds to the multiplication of matrices.

This observation immediately gives us some of the desired properties: (P6) holds for $\mathcal{C}$ (and thus also for $\mathbb{C}$) because matrix multiplication is associative. Similarly, because matrix multiplication distributes over addition, (P9) holds. On the other hand, we do *not* get (P5) for free in this way, because matrix multiplication is in general not commutative. However, if we interchange x_1 and x_2 and also interchange y_1 and y_2 then $x_1x_2 - y_1y_2$ and $x_1y_2 + x_2y_1$ remain unchanged, and thus $z_1 \cdot z_2 = z_2 \cdot z_1$. Thus *these* matrices all commute with each other. As for (P7): the identity matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is $M_{(1,0)}$, so the multiplicative identity in $\mathbb{C}$ is

$$1 := (1, 0) \neq (0, 0) = 0.$$

This also shows (P10). We are left to show that every nonzero element has a multiplicative inverse. This is the only property which requires anything other than direct calculation to establish. Namely, given $z = (x, y) \in \mathbb{C}$ with x and y not both zero, we must find $z^{-1} = (X, Y)$ such that $zz^{-1} = 1$. How do we do that?

Again matrices show us the way. Recall that a two by two matrix

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible iff its determinant $ad - bc$ is nonzero, in which case the inverse is

$$M^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}. \tag{2}$$

(This is a special case of the **adjugate equation**: for any $n \times n$ matrix A, we have $A \cdot \operatorname{adj}(A) = \det A \cdot I_n$.) So if $z = (x, y) \in \mathbb{C}$ with x, y not both zero, then

$$\det M_z = \det \begin{bmatrix} x & -y \\ y & x \end{bmatrix} = x^2 + y^2 \neq 0,$$

so

$$M_z^{-1} = \begin{bmatrix} \frac{x}{x^2+y^2} & \frac{-(-y)}{x^2+y^2} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{bmatrix}.$$

From this we see that

$$z^{-1} = \left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2} \right).$$

We have shown that $\mathbb{C}$, endowed with its usual vector addition and "new" multiplication, is a field.

Above we saw that $(1, 0)$ is the multiplicative identity, so we denote it 1. We now denote the second standard basis vector $(0, 1)$ by i. Thus an arbirary complex number can be written as

$$z = (x, y) = x \cdot 1 + y \cdot i = x + yi.$$

Finally, we observe that

$$i^2 = (0, 1) \cdot (0, 1) = (0 \cdot 0 - 1 \cdot 1, 0 \cdot 1 + 1 \cdot 0) = (-1, 0) = -1(1, 0) = -1.$$

This recovers the standard description of complex numbers, as being the sum of a real number x and a real number y multiplied by an "imaginary number" i satisfying $i^2 = -1$. We could have started here, of course, but our longer route has several advantages. For one, we have made clear "what i is." It seems a little quaint from a modern perspective, but in fact for hundreds of years complex numbers were used by mathematicians but distrusted by them. That i is "imaginary" was something that they worried deeply about: for any magnitude (i.e., real number) x we have $x^2 \geq 0$, so what kind of number can have square -1? The kind of number that does not exist?!? The *representation* complex numbers as points in the plane was introduced by Wessel in 1797 and by Argand in 1806, but mathematicians at that time did not view this as an acceptable *definition* of $\mathbb{C}$: this definition was given by William Rowan Hamilton in 1833. (Hamilton spent a lot of time looking for a product operation on $\mathbb{R}^3$ that makes it into a field. Eventually he realized this is impossible, but he found a multiplication operation on $\mathbb{R}^4$ that satisfies all the field axioms except for the commutativity of multiplication. These are the *quaternions*, and the cross product in $\mathbb{R}^3$ is related to them.)

1.3. **Real part, imaginary part, norm, complex conjugation.** Let

$$z = x + yi \in \mathbb{C}.$$

We define the **real part**

$$\Re(z) = \Re(x + yi) = x$$

and the **imaginary part**

$$\Im(z) = \Im(x + yi) = y.$$

As usual for vectors in the plane, we define the **norm** or **magnitude**

$$|z| = |x + yi| = \sqrt{x^2 + y^2}.$$

Proposition 1.3. *For all $z \in \mathbb{C}$, we have*

$$|\Re(z)| \leq |z|, \ \ |\Im(z)| \leq |z|.$$

Proof. If $z = x + yi$, then

$$|\Re(z)|^2 = x^2 \leq x^2 + y^2 = |z|^2$$

and

$$|\Im(z)|^2 = y^2 \leq x^2 + y^2 = |z|^2.$$

Since the function $x \mapsto x^2$ is increasing on $[0, \infty)$, for non-negative real numbers a, b, we have $a \leq b$ iff $a^2 \leq b^2$. So it follows that

$$|\Re(z)| \leq |z| \text{ and } |\Im(z)| \leq |z|.$$

□

We also define the **complex conjugate**

$$\overline{z} = \overline{x + yi} = x - yi.$$

We also view complex conjugation as a function from $\mathbb{C}$ to $\mathbb{C}$,

$$x + yi \mapsto x - yi.$$

Geometrically, complex conjugation is reflection through the x-axis. In particular it is a linear transformation: for all $\alpha, \beta \in \mathbb{R}$ and $z, w \in \mathbb{C}$, we have

$$\overline{\alpha z + \beta w} = \alpha \overline{z} + \beta \overline{w}.$$

Exercise 1.3. *Let $z, w \in \mathbb{C}$ and $\alpha \in \mathbb{R}$.*
a) Show: $|\alpha z| = |\alpha||z|$.
b) Show: $\overline{\overline{z}} = z$.
c) Show: $|z|^2 = z\overline{z}$.
d) Show: $|\overline{z}| = |z|$.
e) Show: $\overline{zw} = \overline{z}\ \overline{w}$.

f) Show: $|zw| = |z||w|$.
g) Show: if $w \neq 0$, *then* $\overline{\frac{z}{w}} = \frac{\overline{z}}{\overline{w}}$.
h) Show: $\Re(z) = \frac{z+\overline{z}}{2}$ *and* $\Im(z) = \frac{z-\overline{z}}{2i}$.

We've seen complex conjugation already – did you miss it? For nonzero $z = x + yi$, we have

$$z^{-1} = \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2} i = \frac{x - yi}{x^2+y^2}.$$

We can now rewrite this as

$$z^{-1} = \frac{\overline{z}}{|z|^2}. \tag{3}$$

Clearing denominators, we recover Exercise 1.3c). This is a very useful formula: let us give several perspectives. First, it gives another way of finding the inverse of a nonzero complex number: to compute $\frac{1}{z}$, multiply the numerator and the denominator by $\overline{z}$, getting $\frac{\overline{z}}{|z|^2}$.

Example 1.4. *We have*

$$\frac{1}{3+4i} = \frac{3-4i}{3^2+4^2} = \frac{3}{5} - \frac{4}{5}i.$$

It also gives a nice geometric interpretation of z^{-1}: since $z \cdot z^{-1} = 1$, we have

$$1 = |1| = |zz^{-1}| = |z||z^{-1}|,$$

so

$$|z^{-1}| = \frac{1}{|z|}.$$

Thus z^{-1} is obtained from z by reflecting through the origin and then rescaling so that the magnitude is the inverse of the magnitude of z.

Exercise 1.4. *Let* $z \in \mathbb{C}$. *a) Show by induction that for all positive integers* n *we have*

$$|z^n| = |z|^n. \tag{4}$$

b) Explain how to interpret 4 for $n = 0$ *and for negative integers* n, *and show that it still holds.*

As an application of complex conjugation, we give a snappy proof of the triangle inequality in $\mathbb{R}^2$.

Proposition 1.5. *For all* $z, w \in \mathbb{C}$, *we have*

$$|z + w| \leq |z| + |w|. \tag{5}$$

Proof. The first idea is that $|z|^2$ is often easier to work with than $|z|$. Again we have an inequality among non-negative real numbers, so it is enough to prove it after squaring both sides. Now:

$$|z+w|^2 = (z+w)\overline{(z+w)} = z\overline{z} + w\overline{w} + z\overline{w} + \overline{z}w$$
$$= |z|^2 + |w|^2 + 2\Re(z\overline{w}) \leq |z|^2 + |w|^2 + 2|z\overline{w}|$$
$$= |z|^2 + |w|^2 + 2|z||w| = (|z| + |w|)^2.$$

Taking square roots, we get

$$|z + w| \leq |z| + |w|.$$ □

Corollary 1.6. *(Reverse Triangle Inequality) For all* $z, w \in \mathbb{C}$, *we have*

$$||z| - |w|| \leq |z - w|.$$

Proof. By the usual Triangle Inequality we have

$$|z| = |(z-w)+w| \leq |z-w| + |w|,$$

so

$$|z| - |w| \leq |z-w|.$$

Interchanging the roles of z and w gives

$$|w| - |z| \leq |w-z| = |z-w|.$$

Since $||z|-|w||$ is equal to one of $|z|-|w|$ or $|w|-|z|$, we're done. □

1.4. **Geometry of multiplication, polar form and nth roots.** It is important to understand complex numbers not just algebraically but also geometrically (and soon, topologically and analytically: this is a rich subject). So let us begin by geometrically interpreting the field operations.

Again, when it comes to addition this is "something old." The geometric interpretation of addition in $\mathbb{C}$ is as for vectors in $\mathbb{R}^n$ for any n: $z+w$ is understood by taking the tail of w and placing it at the head of w. The relationship between lengths is given by the triangle inequality, whose geometric interpretation is *almost* what it says on the label: the vectors z, w and $z+w$ form three sides of a triangle, and thus the length of $z+w$ does not exceed the lengths of z and w. However, in classical geometry we learn that in fact we have strict inequality when we have an actual triangle: i.e., unless z and w are collinear.

Exercise 1.5. *When does equality hold in $|z+w| \leq |z|+|w|$?*
a) By examining the proof of Proposition 1.5, show that if equality holds, then $\Re(z\overline{w}) = |z||w|$.
b) Show that for complex numbers z,w, we have $\Re(z\overline{w}) = |z||w|$ iff $zw = 0$ (a trivial case) or there is $\alpha \in \mathbb{R}^{>0}$ such that $w = \alpha z$.

Part b) of Exercise 1.5 is rather challenging at the moment. If you don't see how to do it, try it again after learning the very next thing we will discuss!

On to that next thing, namely the "new" geometric interpretation of the product zw of two complex numbers. Again, a good foothold for this is the matrix representation of $z = x+yi$ as

$$M_z = \begin{bmatrix} x & -y \\ y & x \end{bmatrix}.$$

Geometrically then, multiplication by z gives a linear transformation of $\mathbb{R}^2$: $w \mapsto zw$. We should (ideally) have some practice in geometrically interpreting linear transformations of the plane. A good first step is to recall that the columns of the matrix say where the two standard basis $e_1 = 1$, $e_2 = i$ go. The first column is (x_1, y_1), corresponding to the fact that $1 \cdot z_1 = z_1$. The second column is $(-y_1, x_1)$, corresponding to the fact that

$$zi = (x+yi)i = -y + xi.$$

But now we're close enough to the action to notice something: the inner product of the two column vectors (x,y) and (y,x) is zero, meaning that the two vectors are orthogonal ("perpendicular") to each other. Moreover they have the same length, and moreover still $(-y_1, x_1)$ is obtained from (x_1, y_1) via counterclockwise rotation through an angle of $\frac{\pi}{2}$ degrees. Another way to say this is that the matrix corresponding to i is a rotation matrix through an angle of $\frac{\pi}{2}$:

$$M_i = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \cos(\frac{\pi}{2}) & -\sin(\frac{\pi}{2}) \\ \sin(\frac{\pi}{2}) & \cos(\frac{\pi}{2}). \end{bmatrix}.$$

Coming back to the general case, since the columns of M_z are orthogonal to each other, M_z is *almost* an orthogonal matrix. Recall that a matrix is orthogonal iff its columns form an orthonormal basis: they are orthogonal to each other and all have unit length. It is the second condition that is satisfies for $z = i$ – since i lies on the unit circle – but not in general. But okay, we can correct for this just by rescaling: for any $0 \neq z \in \mathbb{C}$, we have

$$M_z = |z| M_{\frac{z}{|z|}} = |z| \cdot \begin{bmatrix} \frac{x}{x^2+y^2} & \frac{-y}{x^2+y^2} \\ \frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{bmatrix}.$$

The matrix $M_{\frac{z}{|z|}}$ is an orthogonal matrix with determinant 1, hence a rotation matrix: there is $\theta \in \mathbb{R}$ such that

$$\begin{bmatrix} \frac{x}{x^2+y^2} & \frac{-y}{x^2+y^2} \\ \frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta. \end{bmatrix}.$$

This gives the following **polar form** of a complex number: every $z \in \mathbb{C}$ can be written as

$$r(\cos\theta + i\sin\theta)$$

for $r \in \mathbb{R}^{\geq 0}$ and $\theta \in \mathbb{R}$. Here $r = |z|$ so is uniquely determined by z. The θ is called the **argument** of z and (use of the definite article "the" notwithstanding!) is *not* uniquely determined: if $z = 0$ then θ can be anything (this is a trivial case). If $z \neq 0$, then – since sine and cosine are periodic with period 2π – if θ is an argument, so is $\theta + 2\pi n$ for any integer n.

Exercise 1.6. *Let $z = x + iy$ be a nonzero complex number. Recall that* $\arctan : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$ *denotes the principal branch of the arctangent function.*
a) Suppose $x > 0$. Show: $\arctan(\frac{y}{x})$ *is an argument for z.*
b) Suppose $x < 0$. Show: $\arctan(\frac{y}{x}) + \pi$ *is an argument for z.*
c) What happens when $x = 0$?

Of course if we rotate through an angle of θ_1 and follow by rotating through an angle of θ_2, the net effect should be that of rotating through an angle of $\theta_1 + \theta_2$. The following exercise asks you to confirm this algebraically.

Exercise 1.7. *Let θ_1 and θ_2 be real numbers. Show:*

$$\begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix} = \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{bmatrix}.$$

Suggestion: first multiply out the matrices, then use the identities

$$\cos(\theta_1+\theta_2) = \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2), \tag{6}$$

$$\sin(\theta_1+\theta_2) = \cos(\theta_1)\sin(\theta_2) + \sin(\theta_1)\cos(\theta_2). \tag{7}$$

Using this exercise, we get a pleasant formula for multiplying complex numbers in polar form: if

$$z = \alpha(\cos\theta_1 + i\sin\theta_1),\ w = \beta(\cos\theta_2 + i\sin\theta_2),$$

then

$$zw = \alpha\beta(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)).$$

In other words, to multiply two complex numbers, we multiply their magnitudes and add their arguments. (Earlier you were asked to show that $|zw| = |z||w|$ for all $z, w \in \mathbb{C}$. We now have a more insightful explanation for this.)

We now introduce "an alternate notation" for $\cos\theta + i\sin\theta$, namely

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

In other words, $e^{i\theta}$ is the complex number lying on the unit circle with angle θ (well-determined up to an integer multiple of 2π). Let me be honest: this is *not just notation*! In fact later we will give an independent definition of e^z and show the identity

$$e^{iz} = \cos z + i \sin z$$

for all $z \in \mathbb{C}$. The following exercise asks you to verify this *formally*, i.e., using power series expansions as in calculus but without (for now!) worrying about convergence.

Exercise 1.8. *Recall the Taylor series expansions*

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!},\ \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!},\ \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!},$$

valid for all $x \in \mathbb{R}$.
a) Show "formally" – i.e., without worrying about convergence – that

$$e^{ix} = \cos x + i \sin x.$$

More precisely, starting with $e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!}$, collect terms with even powers of i to get the expansion for $\cos x$ and collect terms with odd powers of i to get the expansion for $i \sin x$.
b) Show "formally" that for all $x, y \in \mathbb{R}$ we have

$$e^{ix} e^{iy} = e^{i(x+y)}.$$

Do not *show this by using trigonometric identities.*
c) Use a) and b) to get a new proof of the trigonometric identities (6) and (7).

Example 1.7. *We will show that for all $\theta \in \mathbb{R}$ and $\alpha \in \mathbb{R}^{>0}$ we have*

$$\overline{\alpha e^{i\theta}} = \alpha e^{-i\theta}.$$

Indeed, we have

$$\overline{\alpha e^{i\theta}} = \overline{\alpha}\overline{\cos\theta + i \sin\theta} = \alpha(\cos\theta - i\sin\theta) = \alpha e^{i(-\theta)}.$$

Thus complex conjugation preserves the magnitude and negates the argument.

Let $n \in \mathbb{Z}^+$. A complex number z is an **nth root of unity** if $z^n = 1$. A **root of unity** is a complex number that is an nth root of unity for some $n \in \mathbb{Z}^+$. For instance, 1 is an nth root of unity for all $n \in \mathbb{Z}^+$, and it is (clearly!) the only 1st root of unity. Also -1 is an 2nd root of unity. Using (4) we find that if $z^n = 1$ then

$$|z|^n = |z^n| = |1| = 1.$$

Since the only non-negative real number x such that $x^n = 1$ is $x = 1$ (if $0 < x < 1$, then the same holds for x^n for all n; if $x > 1$, then the same holds for x^n for all n), this implies that $|z| = 1$. In other words, every root of unity lies on the unit circle in $\mathbb{C}$. In particular the only real numbers that are roots of unity are ± 1. The number i is a 4th root of unity since $i^4 = (i^2)^2 = (-1)^2 = 1$.

We claim that for any $n \in \mathbb{Z}^+$, there are exactly n complex numbers that are nth roots of unity, namely the numbers $e^{\frac{2\pi i m}{n}}$ for $0 \leq m < n$. To absorb the statement it is well to picture it geometrically: one nth root of unity is given by 1 (take $m = 0$). To get the next nth root of unity, we increment the argument by $\frac{2\pi}{n}$, and then we keep doing this. After we increment the argument n times we get $e^{\frac{2\pi i n}{n}} = e^{2\pi i} = \cos(2\pi) + i \sin(2\pi) = 1$, so we are back where we started. Thus the nth roots of unity form the vertices of a regular n-gon inscribed in the unit circle.

Now that we understand the statement, let's see why it is true. One direction is easy: for any $0 \leq m < n$, we have

$$(e^{\frac{2\pi i m}{n}})^n = e^{\frac{2\pi i m n}{n}} = e^{2\pi m i} = \cos(2\pi m) + i \sin(2\pi m) = 1.$$

For the other direction: we saw that an nth root of unity must be of the form $e^{i\theta}$ for some $\theta \in \mathbb{R}$. Moreover, if $1 = (e^{i\theta})^n = e^{in\theta}$, then $n\theta$ is an argument for 1. A more obvious argument for 1 is 0, and any two arguments differ by an integer multiple of 2π, so there is $M \in \mathbb{Z}$ such that

$$n\theta - 0 = n\theta = 2\pi M,$$

so $\theta = \frac{2\pi M}{n}$ for some $M \in \mathbb{Z}$. Moreover, by division with remainder we can write $M = qn + m$ with $0 \leq r < n$ and then

$$e^{i\theta} = e^{\frac{2\pi i M}{n}} = e^{\frac{2\pi i q n}{n}} \cdot e^{\frac{2\pi i m}{n}} = e^{2\pi i q} \cdot e^{\frac{2\pi i m}{n}} = e^{\frac{2\pi i m}{n}}.$$

Using nth roots of unity, we can find the nth roots of any complex number. First one observation: suppose z, w are complex numbers such that $z^n = w$. Then if ζ is any nth root of unity, we also have $(\zeta z)^n = \zeta^n z^n = 1 \cdot w = w$. So as soon as we find one nth roots of a nonzero complex number w, we have found n of them. Now if w is a nonzero complex number, we write it in polar form:

$$w = \alpha e^{i\theta}.$$

It is then not hard to find one complex number z such that $z^n = w$, namely

$$z = \alpha^{\frac{1}{n}} e^{\frac{i\theta}{n}}.$$

Note that here we are using that any positive *real* number has a unique positive nth root: this follows from the Intermediate Value Theorem and that the function $x \mapsto x^n$ is increasing on $(0, \infty)$. Putting together these observations we find that for any $0 \leq m < n$ then

$$z = \alpha^{\frac{1}{n}} e^{\frac{i(\theta + 2\pi m)}{n}}$$

satisfies $z^n = w$. Thus for every nonzero $w \in \mathbb{C}$ we have found n different nth roots. When $w = 1$ we gave an argument to show that these are all the complex numbers such that $z^n = 1$. It is not hard to modify this argument to work for arbitrary w. We leave that to the reader. A more algebraic approach is suggested below.

Exercise 1.9. *Let F be any field, let $n \in \mathbb{Z}^+$, and let $f(t) = a_n t^n + \ldots + a_1 t + a_0$ be a polynomial with $a_0, \ldots, a_n \in F$ and $a_n \neq 0$ (so f has degree n).*
a) Show: if $\alpha \in F$ is such that $f(\alpha) = 0$, then there is a polynomial $g(t)$ with coefficients in F such that

$$f(t) = (t - \alpha) g(t).$$

(Suggestion: make use of division with remainder for polynomials. In particular, divide f by $t - \alpha$ and observe that the remainder must have degree smaller than the degree of $t - \alpha$ and thus be constant. Now evaluate at α.
b) Use induction to show that there are at most n distinct elements $\alpha \in F$ such that $f(\alpha) = 0$. In other words: a polynomial over any field of degree $n \geq 1$ has no more than n distinct roots.
c) Use part b) to show that for any $w \in F$, there are at most n distinct elements $z \in F$ such that $z^n = w$.

1.5. **Topology of $\mathbb{C}$.** For a complex number a and $r > 0$, we define the **open disk**

$$B_a(r) = \{z \in \mathbb{C} \mid |z - a| < r\}$$

and the **closed disk**

$$\overline{B}_a(R) = \{z \in \mathbb{C} \mid |z - a| \leq r\}.$$

Notoice that $\{z \in \mathbb{C} \mid |z - a| = r\}$ is precsiely the set of points of the plane at distance r from the fixed point a and thus it is a circle of radius r centered at a. The open disk consists of all points on the interior of this circle and the closed disk consists of all points on the interior of this circle and the circle itself.

Using these concepts we can define larger classes of subsets of $\mathbb{C}$, as follows. Let S be a subset of $\mathbb{C}$. We say that $z \in S$ lies in the **interior** of S if for some $r > 0$ we have $B_z(r) \subset S$. That is, not only does S contain the point z but it contains some open disk centered at z. We define the **interior** S° of S to be the set of all points lying in the interior of S: this is a subset of S.

We define the **boundary** ∂S of a subset $S \subset \mathbb{C}$ to be the set of all points $z \in \mathbb{C}$ such that for all $r > 0$, the open disk $B_z(r)$ contains points of S and also of its complement $\mathbb{C} \setminus S$.

For any subset S, the interior S° and the boundary ∂S are disjoint. The set $\mathbb{C} \setminus (S^\circ \cup \partial S)$ is sometimes called the **exterior** of S.

Example 1.8. *Let $S = B_0(1)$ be the open unit disk. I claim that S is equal to its own interior. Indeed, if $z \in S$ then $|z| < 1$ and by the triangle inequality we have $B_z(1 - |z|) \subset B_0(1)$. I claim that the boundary of S is the unit circle. First, let z be any point with $|z| = 1$. Let $r > 0$; we may assume that $r < 1$. Then the point $(1 - \frac{r}{2})z$ has distance $\frac{r}{2} < r$ from z and lies in S, while the point $(1 + \frac{r}{2})z$ has distance $\frac{r}{2} < r$ from z and does not lie in S, so $z \in \partial S$. Finally, if $|z| > 1$, then $B_z(|z| - 1)$ is an open disk centered at z lying entirely in the complement of S.*

Let $S = \overline{B}_0(1)$ be the closed unit disk. The interior and boundary are the same as above: please check! But this time, S° is a proper subset of S whereas $S \supset \partial S$.

Finally, suppose that we have a subset S such that $B_0(1) \subsetneq S \subsetneq \overline{B}_0(1)$: that is, S contains the open unit disk, is contained in the closed unit disk, and contains some but not all points of the boundary circle. Then the interior is still the open unit disk, a proper subset, and the boundary is still the unit circle, which S does not contain.

The preceding example serves to motivate the following definition. We say that a subset $S \subset \mathbb{C}$ is **open** if $S = S^\circ$: equivalently, this means that whenever S contains a point, it contains an open disk about that point. We say that a subset $S \subset \mathbb{C}$ is **closed** if $S \supset \partial S$: that is, it contains its boundary.

Thus we see that open disks are open sets and closed disks are closed sets. (Good!) Note well: a set need *not* be either open or closed. Indeed, as above if we take something strictly between an open disk and the corresponding closed disk then it is neither open nor closed.

Proposition 1.9. *For a subset $S \subset \mathbb{C}$, the following are equivalent.*
(i) The set S is closed.
(ii) The set $\mathbb{C} \setminus S$ is open.

Proof. (i) $\implies$ (ii): Suppose S is closed, and let $z \in \mathbb{C} \setminus S$. We want to show that z lies in the interior of $\mathbb{C} \setminus S$. Suppose not: that means that for all $r > 0$, the open disk $B_z(r)$ is not contained in $\mathbb{C} \setminus S$. In turn this means that for all $r > 0$, the open disk $B_z(r)$ intersects S. By definition then $z \in \partial S$. Since S is closed, we get $z \in S$, contradicting the fact that $z \in \mathbb{C} \setminus S$.
(ii) $\implies$ (i): It is very similar: suppose $\mathbb{C} \setminus S$ is open, and let $z \in \partial S$. If z were not in S then z would lie in the open set $\mathbb{C} \setminus S$ and thus $B_z(r) \subset \mathbb{C} \setminus S$ for some $r > 0$ – but this contradicts the fact that $z \in \partial S$. □

Let $z \in \mathbb{C}$. A subset $S \subset \mathbb{C}$ is a **neighborhood of z** if $z \in S^\circ$. To spell this out, it means that there is $R > 0$ such that the open disk $B_z(R)$ is contained in S. A set is open iff it is a neighborhood of its points.

Let $S \subset \mathbb{C}$. A point $z \in S$ is **isolated** if for some $r > 0$ we have $B_z(r) \cap S = \{z\}$: that is, there is some open disk around z which contains no points other than z than z itself. A point $z \in \mathbb{C}$ is an **accumulation point of S** if for all $r > 0$, the disk $B_z(r)$ contains a point of $S \setminus \{z\}$.

Example 1.10. *a) If S is finite, then $\partial S = S$ and every point of S is isolated.*
c) Let S be an open disk. Then every point of S is an accumulation point and no point of S is a boundary point.

The previous example shows that a boundary point need not be an accumulation point and an accumulation point need not be a boundary point. However:

Proposition 1.11.
a) Let $S \subset \mathbb{C}$, and let $z \in \mathbb{C} \setminus S$. Then $z \in \partial S$ iff z is an accumulation point of S.
b) Thus a subset $S \subset \mathbb{C}$ is closed iff it contains all of its accumulation points.

Proof. a) Suppose $z \in \mathbb{C} \setminus S$. If z were not an accumulation point then there is some $r > 0$ such that $B_r(z) \cap (S \setminus \{z\}) = \varnothing$. But since $z \notin S$, this means that $B_r(z) \cap S = \varnothing$, so z is not a boundary point. Conversely, if z is an accumulation point then every open disk around z contains points of S and – since $z \notin S$ – also points outside of S.
b) Indeed, S is closed iff there is no $z \in \mathbb{C}$ that lies in ∂S but not in S. By part a), this holds iff there is no $z \in \mathbb{C}$ that is an accumulation point of S but does not lie in S – in other words, S contains all its accumulation points. □

An open subset S of $\mathbb{C}$ is **connected** if it is nonempty and is not the disjoint union of two nonempty open subsets. In complex analysis, a **domain** is a connected open subset $U \subset \mathbb{C}$. The name is chosen because whereas in calculus we study functions $f : I \to \mathbb{R}$ for an interval in $\mathbb{R}$, in complex analysis we study functions $f : U \to \mathbb{C}$.

Paths: Let $\gamma : [a, b] \to \mathbb{R}^2$ be a function. Then we can write

$$\gamma(t) = (x(t), y(t))$$

where $x : [a, b] \to \mathbb{R}$ and $y : [a, b] \to \mathbb{R}$ are functions. Equivalently, a function $\gamma : [a, b] \to \mathbb{C}$ can be written in the form

$$\gamma(t) = x(t) + iy(t),$$

i.e., both its real part and imaginary part are functions. This gives us an easy way to discuss continuity and diffrentiability of such functions. We say that γ is continuous iff both $x(t)$ and $y(t)$ are continuous. We say that γ is **differentiable** if both $x(t)$ and $y(t)$ are differentiable, and we say that γ is **smooth** if both $x(t)$ and $y(t)$ are differentiable and for all $t \in [a, b]$ either $x'(t) \neq 0$ or $y'(t) \neq 0$. Equivalently, we have a well-defined, nonzero velocity vector at every point. A path is **piecewise smooth** if it is smooth except for finitely many points. A path is **polygonal** if it consists of straight line segments. A polygonal path is piecewise smooth and is not smooth precisely at the points where we change from one line segment to another line segment of different slope.

A path γ is called **closed** if $\gamma(a) = \gamma(b)$. A path γ is **simple** if for all $c, d \in [a, b)$, we have $\gamma(c) \neq \gamma(d)$. In other words, a simple path does not cross itself except that the initial point and the terminal point are allowed to be the same.

A subset $S \subset \mathbb{C}$ is **path connected** if for all $z, w \in \mathbb{C}$ there is a path $\gamma : [0, 1] \to S$ such that $\gamma(0) = z$ and $\gamma(1) = w$. For instance, open and closed disks are path connected. In fact, they are **convex**: we can get from any point to any other point by a single line segment.

Theorem 1.12. *For a nonempty open subset $U \subset \mathbb{C}$, the following are equivalent:*
(i) U is connected.
(ii) U is path connected.
*(iii) U is **xy-connected**: any two points of U can be connected by a polygonal path each segment of which is either horizontal or vertical.*

Proof. (i) $\implies$ (iii): Fix $z \in U$, and let V_1 be the set of all $w \in U$ such that there is an xy-path – i.e., a path obtained by concatenating horizontal and vertical line segments – $\gamma : [a, b] \to U$ that starts at z and ends at w. Let $V_2 := U \setminus V_1$, so V_1 and V_2 are disjoint subsets of U with union U. We claim that V_1 and V_2 are both open subsets of $\mathbb{C}$. If so, because we have assumed that U is connected and we know that V_1 is nonempty – indeed, taking a constant path at z shows $z \in V_1$ – we must have $V_2 = \varnothing$. This means precisely that U is path connected.

Before establishing that V_1 and V_2 are open we make two simple observations:

• If γ_1 is an xy-path from a to b and γ_2 is an xy-path from b to c, then the concatenation $\gamma_2\gamma_1$ is an xy-path from a to c.

• If B is an open disk in the plane, then any two points $a, b \in D$ can be connected by an xy-path lying entirely in B. Indeed, clearly the center of the disk can be connected to any other point by an xy-path with two line segments, which by the previous observation means that any two points in the disk can be connected by an xy-path with four line segments.[2]

To see that V_1 is open: suppose $w \in V_1$. Since U itself is open, there is some open disk $B_w(\epsilon)$ contained in U. Any point $v \in B_w(\epsilon)$ can be connected to w by an xy-path γ_2. Since $w \in V_1$, there is an xy-path γ_1 from z to w. Concatenating γ_1 and γ_2 gives a path from z to v. This shows that $B_w(\epsilon) \subset V_1$ and thus that V_1 is open. Now suppose $w \in V_2$. The argument is in fact very similar: let $B_w(\epsilon)$ be an open disk centered at w that is contained in U. As above, every point v of this disk can be connected to w via an xy-path γ_2. In fact this implies that there is no xy-path γ_1 from z to v, because then concatenating γ_1 and γ_2 would give an xy-path from z to w, but by definition of V_2 there is no such path. It follows that $B_w(\epsilon) \subset V_2$, so V_2 is open.

(iii) $\implies$ (ii): Since an xy-path is a kind of path, this is immediate.

(ii) $\implies$ (i): (Following [FB]) We say a function $f : U \to \mathbb{C}$ is **locally constant** if for all $z \in U$, there is $\epsilon > 0$ such that $B_z(\epsilon) \subset U$ and the function F is constant on $B_z(\epsilon)$. Every constant function is locally constant. We observe that U is connected if and only if every locally contant function is constant. Indeed, if U is not connected, then it can be partitioned into two open subsets V_1 and V_2. Defining f to be 1 on V_1 and 0 on V_2 gives a locally constant function that is not constant. Conversely, if $f : U \to \mathbb{C}$ is a locally constant function that is not constant, choose $z \in U$, and put

$$V_1 := f^{-1}(\{f(v)\}),\ V_2 := f^{-1}(\mathbb{C} \setminus \{f(v)\}).$$

In other words, V_1 is the set of $w \in U$ such that $f(w) = f(z)$ and V_2 is its complement. Because f is locally constant, V_1 and V_2 are both open sets. The set V_1 is not empty because it contains z. The set V_2 is nonempty because f is not constant. Thus V_1 and V_2 give a partition of U into open sets, so U is disconnected.

Now we assume that U is path connected, and let $f : U \to \mathbb{C}$ be a locally constant function. It suffices to show that f is constant: in other words, given $z_1, z_2 \in U$, we must show that $f(z_1) = f(z_2)$. Let $\gamma : [a, b] \to U$ be a path with $\gamma(a) = z_1$ and $\gamma(b) = z_2$. Consider the function $g : [a, b] \to \mathbb{C}$ given by $g = f \circ \gamma$. Let $t \in [a, b]$. There is $\epsilon > 0$ such that f is constant on $B_{\gamma(t)}(\epsilon)$; since γ is continuous, there is $\delta > 0$ such that for all $s \in [a, b]$ with $|s - t| < \delta$, we have $|\gamma(s) - \gamma(t)| < \epsilon$: in other words, γ maps $(t - \delta, t + \delta)$ into $B_{\gamma(t)}(\epsilon)$. Thus $g = f \circ \gamma$ is constant on $(t - \delta, t + \delta)$. In other words, g is also locally constant, meaning that its real and imaginary parts are each locally contant functions form $[a, b]$ to $\mathbb{R}$. Such a function must have identically zero derivative and thus (by the Mean Value Theorem) be constant. In particular we have

$$f(z_1) = f(\gamma(a)) = g(a) = g(b) = f(\gamma(b)) = f(z_2).$$

So f is constant. □

[2]In fact any two points can be connected by an xy-path with four line segments. Prove this if you want!

Remark 1.13. *a) The definitions of connected and path connected apply in any* topological space. *In any topological space we have that path connectedness implies connectednesss. The standard proof of this uses the facts that an interval on the real line is connected and that the image of a connected space under a continuous function is again connected. The argument given here is a bit different and less general.*
b) From this more general perspective it is quite "lucky" that connected open subsets of $\mathbb{C}$ are path connected. This does not even hold for closed, bounded subsets of $\mathbb{C}$. If you are interested, look up the ***topologist's sine curve****.*

Exercise 1.10. *Let I be an interval on the real line. Use the Mean Value Theorem to show that every locally constant function $f : I \to \mathbb{R}$ is constant, and deduce that I is connected.*

1.6. **Sequences and convergence.** Next we study sequences in $\mathbb{C}$. All of these concepts stem from the notion of **distance** between points. We define the distance between points $w, z \in \mathbb{C}$ as $|z - w|$. This is a special case of the notion of distance between vectors in $\mathbb{R}^N$, namely

$$d(x, y) := ||x - y||.$$

We note in passing that all of the concepts that we define in this section makes sense in $\mathbb{R}^N$ and in fact much more generally.

A **metric space** is a set X together with a **metric function**

$$d : X \times X \to \mathbb{R}^{\geq 0}$$

satisfying the following properties:
(MS1) For all $x, y \in X$, we have $d(x, y) = 0 \iff x = y$.
(MS2) For all $x, y \in X$, we have $d(x, y) = d(y, x)$.
(MS3) For all $x, y, z \in X$, we have $d(x, z) \leq d(x, y) + d(y, z)$.

The idea is that given these axioms, it is reasonable to interpret $d(x, y)$ as the **distance between x and y**. In $\mathbb{R}^n$ we take the (standard Euclidean) metric function to be

$$d(\mathbf{x}, \mathbf{y}) = ||\mathbf{x} - \mathbf{y}||.$$

(MS1) and (MS2) are immediate, and (MS3) follows from the triangle inequality:

$$d(x, z) = ||\mathbf{x} - \mathbf{z}|| = ||(\mathbf{x} - \mathbf{y}) + (\mathbf{y} - \mathbf{z})|| \leq ||\mathbf{x} - \mathbf{y}|| + ||\mathbf{y} - \mathbf{z}|| = d(x, y) + d(y, z).$$

The following concepts from Math 3100 make sense in the context of any metric space: sequences, subsequences, boundedness, convergent sequences and Cauchy[3] sequences. Let us briefly make these definitions in this level of generality, although we will soon restrict to the case of $\mathbb{C}$. Namely:

- A sequence in a metric space (X, d) is just a function $x_\bullet : \mathbb{Z}^+ \to X$. As usual we write it $\{x_n\}_{n=1}^\infty$.
- A subsequence of a sequence is obtained by selecting an infinite subset of $\mathbb{Z}^+$ and restricting to those terms, or more formally, by composing $x_\bullet$ with a strictly increasing function $\mathbb{Z}^+ \to \mathbb{Z}^+$.
- In any metric space (X, d), for $x \in X$ and $r > 0$ we define the **open disk**

$$B^\circ(x, r) = \{y \in X \mid d(x, y) < r\}.$$

[3]Augustin-Louis Cauchy, 1789-1857 was a French mathematician. A really excellent French mathematician: more theorems and concepts bear his name than anyone else. The field of complex analysis is due to him – not quite single-handedly, but close.

That is, it is the set of all points of X whose distance from x is less than r. For $r \geq 0$ we define the **closed disk**

$$B^{\bullet}(x,r) = \{y \in X \mid d(x,y) \leq r\}.$$

That is, it is the set of all points of X whose distance form x is at most r.

Notice that in $\mathbb{R}$ itself, an open disk is just an open interval centered at x and a closed disk is just a closed interval centered at x. However, in $\mathbb{R}^2$, an open disk is the set of points lying inside a circle centered at x of radius r – i.e., a legitimate disk – and an open disk is the set of points lying on or inside a circle centered at x of radius r.

• A subset $S \subset X$ is **bounded** if it is contained in some closed disk.

• Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in the metric space (X,d) and let $x \in X$. We say that the sequence **converges to x** if for all $\epsilon > 0$, there is $N \in \mathbb{Z}^+$ such that for all $n > N$ we have $d(x_n, x_) < \epsilon$. We write this symbolically as

$$x_n \to x.$$

A sequence is **convergent** if it converges to some $x \in X$. Notice that this directly generalizes the definition of convergence in $\mathbb{R}$, in which the distance between x_n and x is given by $|x_n - x|$.

Here is one familiar and easy (but important) result.

Proposition 1.14. *Let $\{x_n\}$ be a sequence in a metric space X. If $L, M \in X$ are such that $x_n \to L$ and $x_n \to M$, then $L = M$.*

Proof. Suppose $L \neq M$, and let $\epsilon := \frac{d(L,M)}{2}$. Then there is $N_1 \in \mathbb{Z}^+$ such that for all $n > N$ we have $d(x_n, L) < \epsilon$: equivalently, we have $x_n \in B_L(\epsilon)$. Similarly there is $N_2 \in \mathbb{Z}^+$ such that for all $n > N$ we have $d(x_n, M) < \epsilon$: equivalently, we have $x_n \in B_M(\epsilon)$. Let $N = \max(N_1, N_2)$. Then for all $n > N$ we have $x_n \in B_L(\epsilon) \cap B_M(\epsilon)$. But our choice of ϵ makes

$$B_L(\epsilon) \cap B_M(\epsilon) = \varnothing,$$

a contradiction! So we must have $L = M$. □

Using sequences we get another characterization of closed sets.

Proposition 1.15. *For a subset S of a metric space (X,d), the following are equivalent:*
(i) S is closed.
(ii) Suppose $\{x_n\}$ is a sequence in S that converges to some $x \in X$. Then $x \in S$.

Proof. (i) $\implies$ (ii): Suppose X is closed, and let $\{x_n\}$ be a sequence in S converging to some $x \in X$. Seeking a contradiction we suppose that x does not lie in S. For all $\epsilon > 0$, $B_x(\epsilon)$ contains x_n for all but finitely many $n \in \mathbb{Z}^+$, so in particular it contains a point of S different from x. Thus x is an accumulation point of S. Since S is closed, this means $x \in S$.
(ii) $\implies$ (i): We will prove the contrapositive: if S is not closed, then there is some $x \in X \setminus S$ that is an accumulation point of S. For all $n \in \mathbb{Z}^+$, let x_n be a point of S such that $d(x_n, x) < \frac{1}{n}$. Then $x_n \to x$. □

• A sequence $\{x_n\}$ in the metric space (X,d) is **Cauchy** if for all $\epsilon > 0$, there is $N \in \mathbb{Z}^+$ such that for all $m, n > N$ we have $d(x_m, x_n) < \epsilon$.

Theorem 1.16. *Let $\{\mathbf{x}_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}^N$ (with the standard Euclidean metric), and let $\mathbf{x} \in \mathbb{R}^n$. The following are equivalent:*
(i) We have $\mathbf{x}_n \to \mathbf{x}$.
(ii) For $1 \leq i \leq N$, the sequence $\{\mathbf{x}_{n,i}\}$ of ith components converges to the ith component x_i of $\mathbf{x}$.

Proof. The main idea is just that for $\mathbf{y} = (y_1, \ldots, y_N) \in \mathbb{R}^N$, we have that $||\mathbf{y}||$ is small iff $|y_i|$ is small for all i. A bit more formally: for any $1 \leq i \leq N$, we have

$$|y_i| \leq \sqrt{y_1^2 + \ldots + y_N^2} = ||\mathbf{y}||.$$

(After squaring both sides, this becomes $y_i^2 \leq y_1^2 + \ldots + y_N^2$.) Conversely, if $|y_i| \leq \epsilon$ for all $1 \leq i \leq N$, then $|y_i|^2 \leq \epsilon^2$ for all i, so

$$|y_1|^2 + \ldots + |y_N|^2 \leq N\epsilon^2$$

and thus

$$||\mathbf{y}|| = \sqrt{|y_1|^2 + \ldots + |y_N|^2} \leq \sqrt{N}\epsilon.$$

Since here N is fixed, this is good enough for us. Now to the formal proof.
(i) $\implies$ (ii): Suppose $\mathbf{x}_n \to \mathbf{x}$, fix $1 \leq i \leq N$, and let $\epsilon > 0$. Then there is $M \in \mathbb{Z}^+$ such that for all $n > M$ we have $||\mathbf{x}_n - \mathbf{x}|| < \epsilon$, and thus

$$|\mathbf{x}_{n,i} - x_i| \leq ||\mathbf{x}_n - \mathbf{x}|| < \epsilon.$$

It follows that $\mathbf{x}_{n,i} \to x_i$.
(ii) $\implies$ (i): Suppose that $\mathbf{x}_{n,i} \to \mathbf{x}_i$ for all $1 \leq i \leq N$. Thus for all $1 \leq i \leq N$ there is $M_i \in \mathbb{Z}^+$ such that for all $n > M_i$ we have $|x_{n,i} - x_i| < \frac{\epsilon}{\sqrt{N}}$. Take $M = \max(M_1, \ldots, M_N)$. Then for all $n > M$ we have

$$||\mathbf{x}_n - \mathbf{x}|| \leq \sqrt{N}\left(\frac{\epsilon}{\sqrt{N}}\right) = \epsilon. \qquad \square$$

Exercise 1.11. *Let $\{\mathbf{x}_n\}_{n=1}^\infty$ be a sequence in $\mathbb{R}^N$. Show that the following are equivalent:*
(i) The sequence $\{\mathbf{x}_n\}$ is bounded.
(ii) For all $1 \leq i \leq qN$, the sequence $\{\mathbf{x}_{n,i}\}$ of ith components is bounded.

Proposition 1.17. *Let (X, d) be a metric space, and let $\{x_n\}$ be a convergent sequence in X. Then $\{x_n\}$ is a Cauchy sequence.*

Proof. We know this result when X is the real numbers with the standard Euclidean metric, and the proof carries over easily. Namely, suppose that $x_n \to x$, fix $\epsilon > 0$, and choose $M \in \mathbb{Z}^+$ such that for all $n > M$ we have $d(x_n, x) < \frac{\epsilon}{2}$. Then if $m, n > N$ we have

$$d(x_m, x_n) \leq d(x_m, x) + d(x, x_n) = d(x_m, x) + d(x_n, x) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \qquad \square$$

Proposition 1.18. *Let (X, d) be a metric space, and let $\{x_n\}$ be a Cauchy sequence in X. Then $\{x_n\}$ is bounded.*

Proof. Let $M \in \mathbb{Z}^+$ be such that if $m, n \geq M$ we have $d(x_m, x_n) < 1$. Let $D = \max_{1 \leq n \leq M} d(x_1, x_n)$. We claim that for all $n \in \mathbb{Z}^+$ we have $d(x_1, x_n) \leq D + 1$ and thus the entire sequence lies in the closed disk of radius $D+1$ centered at x_1. So let $n \in \mathbb{Z}^+$. If $n \leq M$ this is clear from the definition of D. If $n > M$, then we have

$$d(x_1, x_n) \leq d(x_1, x_M) + d(x_M, x_n) \leq D + 1. \qquad \square$$

Proposition 1.19. *Let $\{x_n\}$ be a Cauchy sequence in a metric space (X, d). If $\{x_n\}$ admits a convergent subsequence, then it is itself convergent.*

Proof. Suppose the subsequence $\{x_{n_k}\}_{k=1}^\infty$ converges to x. Fix $\epsilon > 0$. Let $M \in \mathbb{Z}^+$ be such that if $\max(m, n) > M$, then $d(x_m, x_n) < \frac{\epsilon}{2}$. Let $K \in \mathbb{Z}^+$ be such that $n_K \geq M$ and $d(x_{n_K}, x) < \frac{\epsilon}{2}$. If $n > M$, then we have

$$d(x_n, x) \leq d(x_n, x_{n_K}) + d(x_{n_K}, x) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \qquad \square$$

Theorem 1.20. *(Bolzano[4]-Weierstrass[5] in $\mathbb{R}^N$)*
Every bounded sequence in $\mathbb{R}^N$ admits a convergent subsequence.

Proof. The idea is to exploit the Bolzano-Weierstrass Theorem in $\mathbb{R}$ (which we take as known; it is one of the cornerstones of Math 3100) by *repeatedly passing to subsequences.* Namely, let $\mathbf{x}$ be a bounded sequence in $\mathbb{R}^N$. By Exericse 1.11, this means that for all $1 \leq i \leq N$, the sequence $\mathbf{x}_{n,i}$ of ith components is also bounded. So, by Bolzano-Weierstrass in $\mathbb{R}$, there is a subsequence – let's call it $\mathbf{y}_n$ – of $\mathbf{x}_n$ such that the sequence $\mathbf{y}_{n,1}$ of first components converges, say to L_1. Now we move on to to the second component, using $\mathbf{y}_n$ instead of $\mathbf{x}_n$: the sequence $\mathbf{y}_{n,2}$ of second components is bounded, so by Bolzano-Weierstrass in $\mathbb{R}$, there is a subsequence – let's call it $\mathbf{z}_n$ – of $\mathbf{y}_n$ such that $\mathbf{y}_{n,2}$ converges, say to L_2. Since a subsequence of a convergent sequence remains convergent, also $\mathbf{y}_{n,1} \to L_1$. And now we move on to the third component and extract another subseuqence...and so forth. At the end, after passing to a subsequence N times we get a sequence – let's call it ω_n – such that $\omega_{n,i} \to L_i$ for all $1 \leq i \leq N$, and thus $\omega_n \to (L_1, \ldots, L_N)$. The sequence ω_n is still a subsequence of the original sequence $\mathbf{x}$, so we're done. $\square$

Recall that one version of the fundamental completeness property of $\mathbb{R}$ is that every Cauchy sequence in $\mathbb{R}$ is convergent. This is not true in an arbitrary metric space: e.g. it is not true in the rational numbers with the same distance function $d(x, y) = |x - y|$ inherited from $\mathbb{R}^n$. A metric space (X, d) is **complete** if every Cauchy sequence converges.

Theorem 1.21. *The space $\mathbb{R}^n$ endowed with the standard Euclidean metric, $d(\mathbf{x}, \mathbf{y}) = ||\mathbf{x} - \mathbf{y}||$, is a complete metric space.*

Proof. Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $\mathbb{R}^N$. By Proposition 1.18, $\{x_n\}$ is bounded. By Bolzano-Weierstrass, the sequence $\{x_n\}$ admits a convergent subsequence. So $\{x_n\}_{n=1}^{\infty}$ is convergent by Proposition 1.19. $\square$

1.7. **Continuity and limits.** Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is **continuous** at $x \in X$ if: for all $\epsilon > 0$, there is $\delta > 0$ such that for all $x' \in X$, if $d(x, x') < \delta$ then $d(f(x), f(x')) < \epsilon$. A function $f : X \to Y$ is **continuous** if it is continuous at every $x \in X$.

Example 1.22. *Let $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = z^2$. We will show – from scratch – that f is continuous. Let $c \in \mathbb{C}$, and let $\epsilon > 0$. We must find $\delta > 0$ such that if $|z - c| < \delta$ then $|z^2 - c^2| < \epsilon$. Well, we have $|z^2 - c^2| = |z + c||z - c|$. Let us first agree to take $\delta \leq 1$, i.e., $|z - c| \leq 1$. Then (e.g. by the Reverse Triangle Inequality) we have*

$$|z| \leq |c| + 1.$$

So

$$|z + c| \leq |z| + |c| \leq 2|c| + 1,$$

and thus

$$|z^2 - c| = |z + c||z - c| \leq (2|c| + 1)|z - c| < \epsilon$$

iff $|z - c| < \frac{\epsilon}{2|c|+1}$. Thus if we take

$$\delta := \min\left(1, \frac{\epsilon}{2|c| + 1}\right),$$

[4]Bernardus Placidus Johann Nepomuk Bolzano, 1781-1848, was an Italian-Bohemian mathematician, logician, philosopher, theologian and Catholic priest. He had critical early contributions to the field of mathematical analysis, which unfortunately were largely overlooked in his lifetime.

[5]Karl Theodor Wilhelm Weierstrass, 1815-1896, was a German mathematician. He is "the father of modern analysis" – all the fundamental definitions of convergence involving ϵ's are due to him. In the area of *complex* analysis, he is second only to Cauchy. Whereas Cauchy based his theory around contour integrals, Weierstrass emphasized power series expansions. In this course we will recapitulate history: first Cauchy, then Weierstrass.

then $|z-c|<\delta \implies |z^2-c^2|<\epsilon$.

Proposition 1.23. *Let* $f:(X,d_X)\to(Y,d_Y)$ *be a continuous function between metric spaces. Let* $\mathbf{x}$ *be a sequence in* X*. If* $\mathbf{x}$ *converges to* $L\in X$*, then* $f(\mathbf{x})$ *converges to* $f(L)\in Y$*.*

Proof. Let $\epsilon>0$. Then there is $\delta>0$ such that for $x\in X$, if $d(x,L)<\delta$ then $d(f(x),f(L))<\epsilon$. Moreover there is $N\in\mathbb{Z}^+$ such that for $n>N$ we have $d(x_n,L)<\delta$. So, for $n>N$ we have $d(f(x_n),f(L))<\epsilon$. □

Now let (X,d_X) and (Y,d_Y) be metric spaces, and let $x\in X$. We want to define the notion of a function $f:X\setminus\{x\}\to Y$ having a limit at x. We will define this concept in terms of continuity, which is a little simpler and more intuitive. First we need the following observation.

Lemma 1.24. *Let* (X,d_X) *and* (Y,d_Y) *be metric spaces, and let* $x\in X$*. Let* $f:X\setminus\{x\}\to Y$ *be a function.*
a) If x *is an isolated point of* X*, then* f *is continuous at* x *no matter how we define* f *at* x*.*
b) If x *is an accumulation point of* X*, then there is at most one* $y\in Y$ *such that defining* $f(x):=y$ *makes* f *continuous at* x*.*

Proof. a) Let $y\in Y$ and put $f(x):=y$. Since x is isolated in X, there is $\delta>0$ such that $B_x(\delta)=\{x\}$. But this δworks for all ϵ, since if $d(x,x')<\delta$ we must have $x'=x$ and $d(f(x),f(x'))=0<\epsilon$.
b) Suppose that $y\in Y$ is such that putting $f(x):=y$ makes f continuous at x. Since x is an accumulation point of X, there is a sequence $x_n\in X\setminus\{x\}$ such that $x_n\to x$. By Proposition 1.23 it follows that

$$y=f(x)=\lim_{n\to\infty}f(x_n).$$

But now we've won: clearly if we changed y to any other $y'\in Y$, the above would not be true. This completes the proof. □

Now we can give the desired definition of a limit of a function at an accumulation point $x\in X$. Namely, if $f:X\setminus\{c\}\to Y$ is a function, we say that $\lim_{x\to c}f(x)=L$ if defining $f(x):=L$ makes f continuous at x. By Lemma 1.24 there is at most one such L, so the definition works. (Of course there need not be any such L, i.e., the limit need not exist. But this should be familiar even from freshman calculus.)

Okay, we can certainly reformulate this in the usual terms: if $f:X\setminus\{c\}\to Y$ is a function, we say $\lim_{x\to c}f(x)=L$ if: for all $\epsilon>0$, there is a $\delta>0$ such that for all $x\in X\setminus\{c\}$, if $d(x,c)<\delta$ then $d(f(x),f(c))<\epsilon$.

Proposition 1.25. *Let* a *be an accumulation point of the metric space* X *and let* $f:X\setminus\{a\}\to Y$ *be a function. Suppose* $\lim_{x\to a}f(x)=L$*. Then if* $\{x_n\}$ *is any sequence in* $X\setminus\{a\}$ *such that* $x_n\to a$*, then* $f(x_n)\to f(L)$*.*

Proof. Definining $f(a):=L$, we get a function $f:X\to Y$ that is continuous at a. The result now follows from Proposition 1.23. □

Theorem 1.26. *Let* X*,* Y *and* Z *be metric spaces. Let* $f:X\to Y$ *and* $g:Y\to Z$ *be functions. Suppose* f *is continuous at* $a\in X$ *and* g *is continuous at* $f(a)\in Y$*. Then* $g\circ f$ *is continuous at* a*.*

Proof. Let $\epsilon>0$. Since g is continuous at $f(x)$, there is $\eta>0$ such for all $y\in Y$, if $d(y,f(a))<\eta$ then $f(g(y),g(f(a))<\epsilon$. Since f is continuous at a, there is $\delta>0$ such that for all $x\in X$, if $d(x,a)<\delta$ then $d(f(x),f(a))<\eta$. It follows then that if $d(x,a)<\delta$ then $d(g(f(x)),g(f(a)))<\epsilon$. So $g\circ f$ is continuous at a. □

Here is an analogous statement in which we only assume that $\lim_{x\to a}f(x)$ exists.

Theorem 1.27. *Let X, Y and Z be metric spaces, and let $a \in X$ be an accumulation point. Let $f : X \setminus \{a\} \to Y$ and $g : Y \to Z$ be functions. Suppose that $\lim_{xraa} f(x) = L \in Y$ and that g is continuous at L. Then*

$$\lim_{x\to a} g(f(x)) = g(L) = g(\lim_{x\to a} f(x)).$$

Proof. If we define $f(a) := L$ then we can apply the previous result! □

1.8. **The Extreme Value Theorem.**

Theorem 1.28. *Let $A \subset \mathbb{R}^N$ be closed and bounded, and let $f : A \to \mathbb{R}$ be continuous. Then f has minimum and maximum values. That is:*
a) There is $x_m \in A$ such that $f(x_m) \le f(x)$ for all $x \in A$.
b) There is $x_M \in A$ such that $f(x_m) \ge f(x)$ for all $x \in A$.

Proof. Step 1: We show that f is bounded. Seeking a contradiction, we assume not. Then it is either unbounded above, in which case there is a sequence $\{x_n\}$ in A such that $f(x_n) \to \infty$ or it is unbounded below, in which case there is a sequence $\{x_n\}$ in A such that $f(x_n) \to -\infty$ (or both). We may as well assume that f is unbounded above; the other case can be handled very similarly. Since A is bounded, by the Bolzano-Weierstrass Theorem the sequence $\{x_n\}$ admits a convergent subsequence $\{x_{n_k}\}$, say $x_{n_k} \to x$. Since A is closed, we have $x \in A$ by Proposition 1.15. But since a subsequence of a sequence that diverges to ∞ also diverges to ∞, we have on the one hand that

$$f(x_{n_k}) \to \infty,$$

while on the other hand, since $x_{n_k} \to x$ and f is continuous, we have

$$f(x_{n_k}) \to f(x),$$

a contradiction. So indeed f is bounded.
Step 2: Since f is bounded, the supremum of the set $f(A)$ is finite; call it M. We want to show that there is $x_M \in A$ such that $f(x_M) = M$. Again, suppose not; then we can define $g : A \to \mathbb{R}$ by $g(x) := \frac{1}{M - f(x)}$. Since f is continuous, so is g. However, because $f(x)$ takes values arbitrarily close to M, $M - f(x)$ takes values arbitrarily close to zero and thus $g : A \to \mathbb{R}$ is a continuous function that is unbounded above, contradicting Step 1. So there is $x_M \in A$ such that $f(x_M) = M$, and thus $f(x_M) \ge f(x)$ for all $x \in A$. Again a very similar argument shows the existence of $x_m \in A$ such that $f(x_m) \le f(x)$ for all $x \in A$. □

A metric space is called **sequentially compact** if every sequence admits a convergent subsequence. The above proof works to show that if X is sequentially compact and $f : X \to \mathbb{R}$ is continuous, then f has minimum and maximum values.

Exercise 1.12. *Let $A \subset \mathbb{R}^N$ be a subset that is* not *both closed and bounded. Show that there is a continuous function $f : A \to \mathbb{R}$ that is unbounded above. (Suggestion: consider the case in which A is not bounded and A is not closed separately.)*

2. Complex derivatives

2.1. **Limits and continuity of functions $f : A \subset \mathbb{C} \to \mathbb{C}$.** Like $\mathbb{R}$, $\mathbb{C}$ is both a metric space and a field. And like $\mathbb{R}$, these operations play well with each other, which gets calculus off the ground. Here are some familiar facts from calculus with $\mathbb{R}$ replaced by $\mathbb{C}$. The proofs are virtually identical to their real counterparts.

Lemma 2.1. *(Upper and Lower Bounds for Continuous Functions) Let $c \in A \subset \mathbb{C}$ and $f : A \to \mathbb{C}$ be a function that is continuous at c.*
a) For any $\epsilon > 0$, there is a $\delta > 0$ such that for all $z \in A$, if $|z - c| < \delta$ then $|f(z)| \le |f(c)| + \epsilon$.

b) Suppose $f(c) \neq 0$. Then for any $\alpha \in (0,1)$, there is a $\delta > 0$ such that for all $z \in A$, if $|z-c| < \delta$, then $|f(z)| \geq \alpha |f(c)|$.

Proof. a) For any $\epsilon > 0$, there is $\delta > 0$ such that for all $z \in A$, if $|z-c| < \delta$ then $|f(z) - f(c)| < \epsilon$. By the Reverse Triangle Inequality, we have

$$|f(z)| - |f(c)| \leq |f(z) - f(c)| < \epsilon,$$

so

$$|f(z)| \leq |f(c)| + \epsilon.$$

b) There is $\delta > 0$ such that $|z-c| < \delta$ implies $|f(z) - f(c)| < (1-\delta)|f(c)|$. The Reverse Triangle Inequality yields

$$|f(z)| - |f(c)| \leq |f(z) - f(c)| < (1-\delta)|f(c)|,$$

so

$$|f(z)| > |f(c)| - (1-\delta)|f(c)| = \delta |f(c)|.$$

□

Theorem 2.2. *Let $A \subset \mathbb{C}$, let $f, g : A \to \mathbb{C}$, and suppose that f and g are continuous at $c \in A$.*
a) For all $\alpha \in \mathbb{C}$, αf is continuous at c.
b) $f + g$ is continuous at c.
c) fg is continuous at c.
d) If $g(c) \neq 0$, then $\frac{f}{g}$ is continuous at c.

Proof. a) If $\alpha = 0$ then αf is the constant function, which is continuous at all points: take $\delta = 1$ for all ϵ! So suppose $\alpha \neq 0$. Let $\epsilon > 0$. Since f is continuous at c, there is $\delta > 0$ such that for all $z \in A$, if $|z-c| < \delta$ then $|f(z) - f(c)| < \frac{\epsilon}{|\alpha|}$. Thus $|\alpha f(z) - \alpha f(c)| < \epsilon$.
b) Choose $\delta_1 > 0$ such that $|z-c| < \delta_1$ implies $|f(z) - f(c)| < \frac{\epsilon}{2}$. Choose $\delta_2 > 0$ such that $|z-c| < \delta_2$ implies $|g(z) - g(c)| < \frac{\epsilon}{2}$. Let $\delta = \min(\delta_1, \delta_2)$. Then $|z-c| < \delta$ implies $|z-c| < \delta_1$ and $|z-c| < \delta_2$, so

$$|f(z) + g(z) - (f(c) + g(c))| \leq |f(z) - f(c)| + |g(z) - g(c)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

c) There is $\delta_1 > 0$ such that

$$|z-c| < \delta_1 \implies |f(z)| - |f(c)| \leq |f(z) - f(c)| < 1$$

and thus $|f(z)| \leq |f(c)| + 1$. There is $\delta_2 > 0$ such that

$$|z-c| < \delta_2 \implies |g(z) - g(c)| < \frac{\epsilon}{2(|f(c)|+1)}.$$

Finally, there exists $\delta_3 > 0$ such that

$$|z-c| < \delta_3 \implies |f(z) - f(c)| < \frac{\epsilon}{2|g(c)|}.$$

(Here we are assuming that $g(c) \neq 0$. If $g(c) = 0$ then we simply don't have the second term in our expression and the argument is similar but easier.) Taking $\delta = \min \delta_1, \delta_2, \delta_3$, for $|z-c| < \delta$ then $|z-c|$ is less than δ_1, δ_2 and δ_3 so

$$|f(z)g(z) - f(c)g(c)| \leq |f(z)||g(z) - g(c)| + |g(c)||f(z) - f(c)|$$
$$< (|f(c)|+1) \cdot \frac{\epsilon}{2(|f(c)|+1)} + |g(c)| \frac{\epsilon}{2|g(c)|} = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

d) We apply Lemma 2.1b) with $\alpha = \frac{1}{2}$: there is $\delta_1 > 0$ such that $|z-c| < \delta_1$ implies $|g(z)| \geq \frac{|g(c)|}{2}$ and thus also

$$\frac{1}{|g(z)||g(c)|} \leq \frac{2}{|g(c)|^2}.$$

Also there exists $\delta_2 > 0$ such that $|z-c| < \delta_2$ implies $|g(z)-g(c)| < \left(\frac{|g(c)|^2}{2}\right)\epsilon$. Take $\delta = \min(\delta_1, \delta_2)$. Then $|z-c| < \delta$ implies

$$|\frac{1}{g(z)} - \frac{1}{g(c)}| = \left(\frac{1}{|g(z)||g(c)|}\right)|g(z)-g(c)| < \frac{2}{|g(c)|^2}\left(\frac{|g(c)|^2}{2}\right)\epsilon = \epsilon. \qquad \square$$

Theorem 2.3. *Let $f, g : A \to \mathbb{C}$, let $c \in A$ be an accumulation point, and suppose that $\lim_{z\to c} f(z) = L$ and $\lim_{z\to c} g(z) = M$.*
a) For all $\alpha \in \mathbb{C}$, we have $\lim_{z\to c} \alpha f(z) = \alpha L$.
b) We have $\lim_{z\to c} f(z) + g(z) = L + M$.
c) We have $\lim_{z\to c} f(z)g(z) = LM$.
d) If $M \neq 0$, then $\lim_{z\to c} \frac{f(z)}{g(z)} = \frac{L}{M}$.

Proof. This follows from the corresponding statements of Theorem 2.2 upon defining $f(c) := L$ and $g(c) := M$. $\square$

Corollary 2.4. *a) Let $P(z) = a_n z^n + \ldots + a_1 z + a_0$ with $a_0, \ldots, a_n \in \mathbb{C}$ be any polynomial. Then P defines a continuous function from $\mathbb{C}$ to $\mathbb{C}$.*
b) Let $R(z) = \frac{P(z)}{Q(z)}$ be a rational function – i.e., a quotient of two polynomials with Q not the zero polynomial. Let $Z := \{z \in \mathbb{C} \mid Q(z) = 0\}$ – a finite set. Then R defines a continuous function from $\mathbb{C} \setminus Z$ to $\mathbb{C}$.

Proof. Certainly the function $f(z) = z$ is continuous on $\mathbb{C}$: take $\delta = \epsilon$. The result now follows immediately from Theorem 2.3 $\square$

Ho hum. No, but actually **WAKE UP!!** Notwithstanding these banal formalities, in many ways limits in $\mathbb{C}$ are more interesting than limits in $\mathbb{R}$. The following example serves to illustrate this.

Example 2.5.
a) Let $f(z) = \overline{z}$, viewed as a function from $\mathbb{C}$ to $\mathbb{C}$. Then f is continuous. In fact it is distance preserving (a.k.a.: an ***isometry****) and all such maps are continuous with $\epsilon = \delta$: if $|z-c| < \epsilon$ then*

$$|\overline{z} - \overline{c}| = |\overline{z-c}| = |z-c| < \epsilon.$$

b) Consider the function $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ given by $f(z) = \frac{\overline{z}}{z}$. It follows from part a) and Theorem 2.2 that f is continuous. However, what about $\lim_{z\to 0} f(z)$? We observe that for all $z \in \mathbb{C} \setminus \{0\}$ we have $|f(z)| = |\frac{\overline{z}}{z}| = \frac{|\overline{z}|}{|z|} = 1$, so f does not "blow up" as we approach 0. Nevertheless the limit does not exist. To see this, first suppose that z is real. Then $f(z) = 1$. In particular, consider the sequence $x_n = \frac{1}{n}$. Then $x_n \to 0$ and each x_n is real, so $f(x_n) = 1 \to 1$. Now suppose that $z = ix$ for x real. Then $\overline{z} = \overline{ix} = -ix = -z$, so $f(z) = -1$. So consider the sequence $y_n = \frac{i}{n}$. Then $y_n \to 0$ and $f(y_n) = -1 \to -1$. This shows that $\lim_{z\to 0} f(z)$ does not exist: if it did, then by Proposition 1.25, for any two sequences x_n and y_n converging to zero, $f(x_n)$ and $f(y_n)$ would converge to the same limit.

There is a deeper moral to be extracted from the above example. In calculus one decomposes the notion of a limit into "left hand limits" and "right hand limits": this gives a sense in which there are "two ways for x to approach c in $\mathbb{R}$". In the complex plane there are infinitely many ways to approach c, because there are infinitely many slopes of lines in the plane.

(Actually, even this is not enough.

Exercise 2.1. *Let $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ be given by $f(x+iy) = \frac{x^2 y}{x^4+y^2}$.*
a) Show that along any line through 0, f approaches 0.
b) Show that along the parabola $y = x^2$, f approaches $\frac{1}{2}$.

But for differentiable functions, this moral will hold true, as we will soon see.)

2.2. **Complex derivatives.** Let $A \subset \mathbb{C}$, and let $c \in A^\circ$. Let $f : A \to \mathbb{C}$. We say that f is **differentiable at c** if

$$\lim_{z \to c} \frac{f(z) - f(c)}{z - c}$$

exists. If so, we denote the value by $f'(z)$ and call it the **derivative** of f.

Example 2.6. *(Quelle surprise)*
a) The function $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = z$ is differentiable at all points, and $f'(z) = 1$. We have

$$f'(c) = \lim_{z \to c} \frac{z - c}{z - c} = \lim_{z \to c} 1 = 1.$$

b) The function $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = z^2$ is differentiable at all points, and $f'(z) = 2z$. Indeed, we have

$$f'(c) = \lim_{z \to c} \frac{z^2 - c^2}{z - c} = \lim_{z \to c} \frac{(z + c)(z - c)}{z - c} = \lim_{z \to c} z + c = 2c.$$

Notice that in each case we have used the following (easiest?) method for evaluating a derivative: namely, simplify the difference quotient until it is clear how to extend it to a continuous function at $z = c$. Then the limit is obtained by evaluating at c.
c) Let $n \in \mathbb{Z}^+$. The function $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = z^n$ is differentiable at all points, and $f'(z) = nz^{n-1}$. Indeed, we have

$$f'(c) = \lim_{z \to c} \frac{z^n - c^n}{z - c} = \lim_{z \to c} \frac{(z - c)(z^{n-1} + z^{n-2}c + \ldots + zc^{n-2} + c^{n-1})}{z - c}$$
$$= \lim_{z \to c} z^{n-1} + z^{n-2}c + \ldots + zc^{n-2} + c^{n-1} = c^{n-1} + \ldots + c^{n-1} = nc^{n-1}.$$

Example 2.7. *Consider $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = \overline{z}$. We claim f is* not *differentiable at 0. Indeed,*

$$f'(0) = \lim_{z \to 0} \frac{\overline{z} - \overline{0}}{z - 0} = \lim_{z \to 0} \frac{\overline{z}}{z},$$

and this is the limit that we showed did not exist in Example 2.5b).

Exercise 2.2. *Adapt the method of Examples 2.5b) and 2.7 to show that $f(z) = \overline{z}$ is not differentiable at any $a \in \mathbb{C}$.*

Now again we have a rather expected result.

Theorem 2.8. *Suppose that $f : A \to \mathbb{C}$ and $g : A \to \mathbb{C}$ be functions. Let $c \in A^\circ$, and suppose that f and g are differentiable at c. Also let $h : B \to \mathbb{C}$ be a function with $g(c) \in B^\circ$, and suppose that h is differentiable at $g(c)$. Then:*
a) For all $\alpha \in C$, αf is differentiable at c and $(\alpha f)'(c) = \alpha f'(c)$.
b) The function $f + g$ is differentiable at c and $(f + g)'(c) = f'(c) + g'(c)$.
c) (Product Rule) The function fg is differentiable at c and

$$(fg)'(c) = f'(c)g(c) + f(c)g'(c).$$

d) (Quotient Rule) If $g(c) \neq 0$, then $\frac{f}{g}$ is differentiable at c and

$$\left(\frac{f}{g}\right)' = \frac{g(c)f'(c) - f(c)g'(c)}{g(c)^2}.$$

e) (Chain Rule) The function $h \circ g$ is differentiable at c and

$$(h \circ g)'(c) = h'(g(c))g'(c).$$

Proof. The proofs are virtually identical to the real variable case: cf. e.g. [C-HC, §5.2]. □

We will also want the following variant of the Chain Rule for paths in $\mathbb{C}$.

Theorem 2.9. *(Chain Rule for Paths) Let $h : B \to \mathbb{C}$ be a function, and let $\gamma : [a, b] \to B$ be a smooth path. If for $t_0 \in [a, b]$ we have that h is differentiable at $\gamma(t_0)$, then the composite function $h \circ \gamma : [a, b] \to \mathbb{C}$ is differentiable at t_0 and*

$$(h \circ \gamma)'(t_0) = h'(\gamma(t_0))\gamma'(t_0).$$

Exercise 2.3. *Prove Theorem 2.9.*
(Suggestion: check that the proof of Theorem 2.8e) carries over essentially verbatim.)

If f is differentiable at all points on some open disk $B_c(\epsilon)$, then we say that f is **holomorphic** at c.

These notions are technically not the same.

Exercise 2.4. *Let $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = \overline{z}^2$.*
a) Show: f is differentiable at 0.
b) Show: f is not differentiable at any other point of $\mathbb{C}$.
c) Conclude: f is not holomorphic at 0.

In practice it will be the holomorphic functions that we care about (and examples like the above will not really arise for us).

A function $f : \mathbb{C} \to \mathbb{C}$ is **entire** if it is holomorphic at every point of $\mathbb{C}$ (equivalently, differentiable at every point of $\mathbb{C}$).

Exercise 2.5. *a) Show that every polynomial $P(z) = a_n z^n + \ldots + a_1 z + a_0$ is an entire function.*
b) Show that every rational function $R(z) = \frac{P(z)}{Q(z)}$ (where P and Q are polynomial functions of z) is holomorphic on $\mathbb{C} \setminus Z$, where $Z = \{z \in \mathbb{C} \mid Q(z) = 0\}$.
c) Must every entire holomorphic function be a polynomial?

Theorem 2.10. *(Inverse Function Theorem) Let $U, V \subset \mathbb{C}$ be open sets, and let $f : U \to V$ be a bijection with inverse function g. Let $d \in V$. If f is differentiable at $g(d)$, $f'(g(d)) \neq 0$ and g is continuous at d, then g is differentiable at d with*

$$g'(d) = \frac{1}{f'(g(d))}.$$

Proof. Put $c \coloneqq g(d)$ and

$$\varphi(w) \coloneqq \begin{cases} \frac{f(w)-f(c)}{w-c} & w \neq c \\ f'(c) & f = c \end{cases}.$$

Then φ is continuous at c. Using the identity $f(g(z)) = z$ for all $z \in V$, we get

$$g'(d) = \lim_{z \to d} \frac{g(z) - g(d)}{z - d} = \lim_{z \to d} \frac{g(z) - g(d)}{f(g(z)) - f(g(d))} = \lim_{z \to d} \frac{1}{\frac{f(g(z))-f(g(d))}{g(z)-g(d)}}$$

$$= \lim_{z \to d} \frac{1}{\varphi(g(z))} = \frac{1}{\varphi(\lim_{z \to d} g(z))} = \frac{1}{f'(c)} = \frac{1}{f'(g(d))}.$$ □

Remark 2.11. *In fact, if $U \subset \mathbb{C}$ is open and $f : U \to \mathbb{C}$ is continuous and injective, then $V \coloneqq f(U)$ is necessarily open and the inverse function $g : V \to U$ is necessarily continuous. But this is a deep theorem of L.E.J. Brouwer called **Invariance of Domain**. (Brouwer's Theorem is valid for $\mathbb{R}^N$ in place of $\mathbb{C}$.)*

Here is an interesting geometric property of holomorphic mappings.

Proposition 2.12. *Let $f : U \to \mathbb{C}$ be holomorphic at $a \in \mathbb{C}$ with $f'(a) \neq 0$. Let γ_1, γ_2 be two smooth paths in $\mathbb{C}$ passing through a, such that the tangent vectors make an angle of θ at a. Then $f \circ \gamma_1$ and $f \circ \gamma_2$ are smooth paths passing through $f(a)$ such that the tangent vectors make an angle of θ at a.*

Proof. Although we could go ahead with the proof, let's work up to it by first considering the case that $f(z) = \alpha z + b$ is a linear map. Then certainly f is holomorphic and the derivative is $f'(z) \equiv \alpha$, so the hypothesis is that $\alpha \neq 0$. Moreover, the map $w \mapsto w + b$ is just a translation in the plane, hence certainly preserves angles so the basic case is $f(z) = \alpha z$, and it comes down to the geometry of multiplication...which we already know. Writing $\alpha = re^{i\theta}$ we express f as the composition of the rotation $z \mapsto e^{iz}$ and the dilation $z \mapsto rz$. Both of these maps preserve angles. The idea behind the general case is that the derivative is the linear approximation to the map f, and this should be enough to imply that since a linea map preserves angles, so does f.

Now we pursue the general case. Suppose $\gamma_1, \gamma_2 : [0, 1] \to \mathbb{C}$ are smooth paths with $\gamma_i(0) = a$ for $i = 1, 2$. So the tangent vectors in question are $\gamma_1'(0)$ and $\gamma_2'(0)$; suppose they make an angle of θ. Now Theorem 2.9 gives that for $i = 1, 2$ we have

$$\frac{d}{dt} f(\gamma_i(t))|_{t=0} = f'(\gamma_i(0))\gamma_i'(0) = f'(a)\gamma_i'(0).$$

In other words, the two tangent vectors after applying f are precisely $f'(a)$ times the two tangent vectors before applying f. So we have reduced the fact that multiplication by a nonzero complex number preserves angles. □

A differentiable function $f : \mathbb{R}^2 \to \mathbb{R}^2$ that preserves angles between tangent vectors is called **conformal** or a **conformal mapping**. Thus Proposition 2.12 can be succinctly restated by stating that a holomorphic function with nonvanishing derivative is conformal.

2.3. The Cauchy-Riemann Equations I.

Recall that for a function $f : \mathbb{R}^2 \to \mathbb{R}$ we have defined **partial derivatives**

$$\frac{\partial f}{\partial x}(x_0, y_0) := \lim_{x \to x_0} \frac{f(x, y_0) - f(x_0, y_0)}{x - x_0}, \ \frac{\partial}{\partial y}(x_0, y_0) := \lim_{y \to y_0} \frac{f(x_0, y) - f(x_0, y_0)}{y - y_0}.$$

That is: to define $\frac{\partial f}{\partial x}$, we fix $y = y_0$ and take the derivative as a function of x alone, whereas to define $\frac{\partial f}{\partial y}$ we fix $x = x_0$ and take the derivative of a function of y alone.

Now let $f : \mathbb{C} \to \mathbb{C}$. Since $\mathbb{C} = \mathbb{R}^2$ with some extra structure, we can regard $f : \mathbb{R}^2 \to \mathbb{C}$ and thereby define partial derivatives in exactly the same way, with the sole difference being that we now get a complex number rather than a real number:

$$\frac{\partial f}{\partial x}(x_0, y_0) := \lim_{x \to x_0} \frac{f(x, y_0) - f(x_0, y_0)}{x - x_0}, \ \frac{\partial}{\partial y}(x_0, y_0) := \lim_{y \to y_0} \frac{f(x_0, y) - f(x_0, y_0)}{y - y_0}.$$

But it may be helpful to spell it out: writing $f(z) = u(z) + iv(z)$, this amounts to

$$\frac{\partial f}{\partial x}(x_0, y_0) := \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0),$$
$$\frac{\partial f}{\partial y}(x_0, y_0) := \frac{\partial u}{\partial y}(x_0, y_0) + i\frac{\partial v}{\partial y}(x_0, y_0).$$

The main question that launches this section is: if $f : \mathbb{C} \to \mathbb{C}$, what is the relationship between f being differentiable in the complex sense and the existence of the partial derivatives $\frac{df}{dx}$ and $\frac{df}{dy}$

– equivalently, of $\frac{du}{dx}$, $\frac{du}{dy}$, $\frac{dv}{dx}$, $\frac{dv}{dy}$? There is a beautiful and useful answer given by the **Cauchy-Riemann[6] equations**. It comes in two parts, so first things first.

Theorem 2.13. *(Cauchy-Riemann Equations, Part I) Let $A \subset \mathbb{C}$, let $z_0 = x_0 + iy_0 \in A$ be an accumulation point, and let $f : A \to \mathbb{C}$ be a function. If f is differentiable at a, then $\frac{df}{dx}$ and $\frac{df}{dy}$ both exist and we have*

$$\frac{df}{dx}(z_0) = -i\frac{df}{dy}(z_0). \tag{8}$$

Proof. What we are assuming is that $f'(z_0) = \lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists. The sole idea of the proof is to take advantage of the fact that we are allowed to approach z_0 in both the x and y directions. Namely, first take z of the special form

$$z = x + iy_0.$$

Then

$$f'(z) = \lim_{x+iy_0 \to z_0} \frac{f(x+iy_0) - f(z_0)}{(x+iy_0) - (x_0+iy_0)} = \lim_{x \to x_0} \frac{f(x,y_0) - f(x_0,y_0)}{x - x_0} = \frac{\partial f}{\partial x}(z_0).$$

(In the penultimate inequality we have identified $x + iy$ with (x, y)...as of course we did at the beginning of the course.) Now we take z of the special form $z = x_0 + iy_0$. Then

$$f'(z) = \lim_{x_0+iy \to z_0} \frac{f(x_0+iy) - f(z_0)}{(x_0+iy) - (x_0+iy_0)} = \lim_{y \to y_0} \frac{f(x_0,y) - f(x_0,y_0)}{i(y - y_0)}$$
$$= \frac{1}{i}\frac{df}{dy}(z_0) = -i\frac{df}{dy}(z_0).$$

So

$$\frac{df}{dx}(z_0) = f'(z_0) = -i\frac{df}{dy}(z_0).$$

□

The Cauchy-Riemann equations can be rewritten in several convenient forms. First we rewrite them in the form that is most traditional: taking $f(z) = u(z)+iv(z)$ as above, then we can replace the one complex equation to two real equations:

$$\frac{du}{dx} + i\frac{dv}{dx} = \frac{df}{dx} = -i\frac{df}{dy} = -i(\frac{du}{dy} + i\frac{dv}{dy}) = \frac{dv}{dy} - i\frac{du}{dy}.$$

Equating real and imaginary parts we get that if f is differentiable at z_0 then

$$\frac{du}{dx}(z_0) = \frac{dv}{dy}(z_0),\ \frac{dv}{dx}(z_0) = -\frac{du}{dy}(z_0). \tag{9}$$

Now we give yet another way to write partial derivatives, which it is perhaps best to think of as simply being a different coordinate system from x and y. Namely, we define

$$\frac{\partial f}{\partial z} := \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right),$$
$$\frac{\partial f}{\partial \overline{z}} := \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

[6]Georg Friedrich Bernhard Riemann, 1826-1866, was a visionary German mathematician. After Cauchy and Weierstrass he is the third founder of complex analysis. From a contemporary perspective, his *geometric* approach – associating topological spaces to multivalued analytic functions – is perhaps the most important. Unfortunately his ideas are hard to convey at the undergraduate level, and the geometric aspects of our course will be intermittent and incomplete, though perhaps still suggestive.

(You can "derive" these from the chain rule if you assume that $\frac{\partial z}{\partial \overline{z}} = \frac{\partial \overline{z}}{\partial z} = 0$.) In this notation, Theorem 2.13 says that if f is complex differentiable at z_0 then

$$\frac{\partial f}{\partial \overline{z}}(z_0) = 0, \ \frac{\partial f}{\partial \overline{z}} = f'(z).$$

Exercise 2.6. *Show that expressed in polar coordinates, the Cauchy-Riemann equations for $f(z) = u(z) + iv(z)$ read as follows:*

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \ \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}.$$

Recall the "Zero Velocity Theorem": that if $I \subset \mathbb{R}$ is an interval and $f : I \rightarrow \mathbb{R}$ is differentiable with $f' \equiv 0$, then f is constant. (In other words: if your instantaneous velocity is zero at any point, then you are not moving.) Here the fact that the domain is an interval is needed for the result to be true. For instance, let $A = (-\infty, 0) \cup (0, \infty)$. Then the function

$$f : D \rightarrow \mathbb{R}, x \mapsto \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$$

is differentiable on D and has $f'(x) = 0$ for all $x \in D$, but is not constant. Rather it is only "locally constant" – i.e., it is constant in an open set around each point. It is a fact of topology that a locally constant function on a connected space is actually constant. We will use a form of this idea to prove the following complex analogue.

Theorem 2.14. *(Complex Zero Velocity Theorem) Let $D \subset \mathbb{C}$ be a connected open set, and let $f : D \rightarrow \mathbb{C}$. Suppose f is differentiable and that $f'(z) = 0$ for all $z \in D$. Then f is constant.*

Proof. Write $f(z) = u(z) + iv(z)$ as usual. Then for all $z \in D$ we have

$$0 = f'(z) = \frac{df}{dx} = \frac{du}{dx} + i\frac{dv}{dv},$$

meaning that $\frac{du}{dx}(z) = \frac{dv}{dx}(z) = 0$. By the Cauchy-Riemann equations, this also implies that $\frac{dv}{dy}(z) = \frac{du}{dx}(z) = 0$. We are therefore reduced to the following real variable statement: if $D \subset \mathbb{R}^2$ is a connected open set and $g : D \rightarrow \mathbb{R}$ is a function such that $\frac{dg}{dx}$ and $\frac{dg}{dy}$ both exist and are identically zero, then g is constant.

Indeed, fixing y and letting x vary, the real variable Zero Velocity Theorem shows that since $\frac{dg}{dx} \equiv 0$, g is constant along each horizontal line. Similarly, fixing x and letting y vary, since $\frac{dg}{dy} \equiv 0$, g is constant along each vertical line. By Theorem 1.12, any two points in D can be connected by a path consisting of horizontal and vertical line segments, so it follows that g is constant. $\square$

Exercise 2.7. *Let $U \subset \mathbb{C}$ be a connected open set, and let $f : U \rightarrow \mathbb{C}$. Suppose that for some $n \in \mathbb{Z}^+$, the function f is n times differentaible and $f^{(n)} = 0$ for all $z \in D$. Show that f is given by a polynomial of degree at most n.*

Proposition 2.15. *Let $U \subset \mathbb{C}$ be a domain, and let $f : U \rightarrow \mathbb{C}$ be holomorphic. If $f(z) \in \mathbb{R}$ for all $z \in U$, then f is constant.*

Proof. Writing $f(z) = u(z) + iv(z)$, our hypothesis on f is precisely that $v = 0$ on U. It follows of course that $\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$, and thus by the Cauchy-Riemann equations we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0, \ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -0 = 0.$$

As in the proof of Theorem 2.14, this implies that u is constant. $\square$

Exercise 2.8. *Let $\ell \subset \mathbb{C}$ be a line, let $U \subset \mathbb{C}$ be a domain, and let $f : U \to \mathbb{C}$ be a holomorphic function such that $f(U) \subset \ell$. Show: f is constant. (Hint: there is $\alpha \in \mathbb{C}^\times$ and $\beta \in \mathbb{C}$ such that putting $g(z) := \alpha z + \beta$, we have that $\{g(z) \mid z \in \ell\} = \mathbb{R}$. Apply Proposition 2.15 to $g \circ f$.)*

2.4. **Harmonic functions.** Let $U \subset \mathbb{C}$ be a domain. A function $f : U \to \mathbb{R}$ is **harmonic** if it has continuous second order partial derivatives and satisfies the **Laplace equation** on U:

$$\Delta(f) := \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \equiv 0.$$

Let us introduce an abbreviated notation for higher partial derivatives, e.g.

$$f_x x := \frac{\partial^2 f}{\partial x^2},\ f_{xy} = \frac{\partial^2 f}{\partial x \partial y}$$

and so forth. So

$$\Delta(f) := f_x x + f_y y.$$

The Laplace equation is probably the single most important example of a **partial differential equation (PDE)**, and harmonic functions are ubiquitous throughout pure and applied mathematics. Well, another one of the most famous examples of PDEs is the Cauchy-Riemann equations, and in fact the two are related. Suppose $f(z) = u(z) + iv(z) : \mathbb{C} \to \mathbb{C}$ is entire. (We could also work with functions defined on a **domain** $U \subset \mathbb{C}$, but for now we just want to hit the most basic point.) Then at all points $z \in \mathbb{C}$, we have $\frac{du}{dx} = \frac{dv}{dy}$ and $\frac{dv}{dx} = -\frac{du}{dy}$. Then since continuity of the second partial derivatives ensures that $f_{xy} = f_{yx}$, we have

$$u_{xx} = (u_x)_x = (v_y)_x = v_{yx} = v_{xy} = (v_x)_y = (-u_y)_y = -u_{yy}$$

and thus

$$\Delta(u) = u_{xx} + u_{yy} \equiv 0.$$

That is, the real part of an entire function (with continuous second partials) is a harmonic function. Similarly,

$$v_{xx} = (v_x)_x = (-u_y)_x = -u_{yx} = -u_{xy} = -(u_x)_y = -(v_y)_y = -v_{yy},$$

so

$$\Delta(v) = v_{xx} + v_{yy} \equiv 0$$

and the imaginary part v of an entire function is also harmonic. Later we will see that for every gunction $u : \mathbb{R}^2 \to \mathbb{R}$ there is another harmonic function $v : \mathbb{R}^2 \to \mathbb{R}$ – uniquely determined up to a constant – such that $f := u+iv$ is entire. (This is related to the multivariable calculus phenomenon of a conversative vector field $F : \mathbb{R}^2 \to \mathbb{R}^2$ being the gradient of a function – both cases are proved via line integration.) We say that u and v are **harmonic conjugates**.

For domains D other than $\mathbb{C}$ itself, harmonic conjugates need not exist: this turns out to be related to the topology of D. (Very roughly speaking, harmonic conjugates exist if D "has no holes.") On the other hand, the essential *uniqueness* of harmonic conjugates is much easier and will be established now.

Theorem 2.16. *Let $D \subset \mathbb{C}$ be an open connected set, and let $u, v_1, v_2 : D \to \mathbb{C}$ be harmonic functions, and put*

$$f_1(z) := u(z) + iv_1(z),\ f_2(z) := u(z) + iv_2(z).$$

If f_1 and f_2 are both analytic on D, then $v_2 - v_1$ is constant.

Proof. Since f_1 and f_2 are both analytic, so is $\frac{f_1 - f_2}{i} = v_1 - v_2$. But $v_1 - v_2$ is real-valued, so Proposition 2.15 applies. □

In other words, if a harmonic function $u : D \to \mathbb{C}$ has any harmonic conjugate at all, then any two harmonic conjugates differ by a constant. (Conversely, if $v : D \to \mathbb{C}$ is any harmonic conjugate of u then so is $v + \alpha$ for any $\alpha \in \mathbb{C}$.)

2.5. **The Cauchy-Riemann Equations II.** It turns out that there is an essential converse to Theorem 2.13, as follows.

Theorem 2.17. *Let $A \subset \mathbb{C}$, let $a \in A^\circ$, and let $f : A \to \mathbb{C}$. Suppose there is $\epsilon > 0$ such that for all $z \in B_a(\epsilon)$ the partial derivatives $\frac{df}{dx}$ and $\frac{df}{dy}$ exist, are continuous and satisfy the Cauchy-Riemann equations: $\frac{df}{dx} = -i\frac{df}{dy}$. Then f is differentiable at a.*

Proof. First Proof: See [BMPS, §2.3].
Second Proof: We will make use of the following definition of the **total derivative** of a function $F : \mathbb{R}^n \to \mathbb{R}^m$ at a point $x_0 \in \mathbb{R}^n$: it is the (unique, if it exists) $m \times n$ matrix $D(F)(x_0)$ such that: for all $\epsilon > 0$, there is $\delta > 0$ such that $0 < ||x - x_0|| < \delta$ implies

$$||f(x) - f(x_0) - D(F)(x_0)(x - x_0)|| < \epsilon||x - x_0||.$$

If a function is differentiable in this sense, then the (i, j)th entry of the matrix $D(F)(x_0)$ must be the corresponding partial derivative $\frac{\partial F_i}{\partial x_j}$ so that $D(F)(x_0)$ is the usual Jacobian matrix of partial derivatives. Moreover, if each partial derivative exists *and is continuous*, then the Jacobian matrix serves as the total derivative at x_0. For more on this, see e.g. [Ma, Thm. 6.4].

Thus in the context of our $f = u + iv : A \to \mathbb{C}$, our hypotheses on continuity of the partial derivatives imply that the total derivative of F is the Jacobian matrix $\begin{bmatrix} \frac{du}{dx} & \frac{du}{dy} \\ \frac{dv}{dx} & \frac{dv}{dy} \end{bmatrix}$. If moreover the Cauchy-Riemann equations apply, then this Jacobian matrix takes the form $\begin{bmatrix} \frac{du}{dx} & -\frac{dv}{dx} \\ \frac{dv}{dx} & \frac{du}{dx} \end{bmatrix}$. We notice that this is precisely a matrix of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ that corresponds to multiplication by the complex number $a + ib$. Thus the condition of the existence of the total derivative can be rewritten in complex notation as: for all $\epsilon > 0$, there is $\delta > 0$ such that $0 < |z - z_0| < \delta$ implies

$$\big|f(z) - f(z_0) - \left(\frac{du}{dx}(z_0) + i\frac{dv}{dx}(z_0)(z - z_0)\right)\big| < \epsilon|z - z_0|,$$

or

$$\frac{|f(z) - f(z_0) - (z - z_0)(\frac{df}{dx})(z_0))|}{|z - z_0|} < \epsilon.$$

This means that

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} - \frac{df}{dx}(z_0) = 0,$$

or equivalently,

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \frac{df}{dx}(z_0).$$

That is, f is differentiable at z_0 and its derivative is $\frac{df}{dx}(z_0)$. $\square$

In other words, a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ that is sufficiently differentiable in the real sense is differentiable in the complex sense iff f satisfies the Cauchy-Riemann equations. Thus we can construct holomorphic functions by solving PDEs.

3. Some complex functions

3.1. **Exponentials, trigonometric and hyperbolic functions.** We begin with a somewhat unusual application of Theorem 2.17. We define the exponential function

$$z \mapsto e^z : \mathbb{C} \to \mathbb{C}$$

as follows: if $z = x + iy$ we put

$$e^z := e^x(\cos y + i \sin y).$$

This is very unlikely looking: a bit of this and a bit of that. Nevertheless:

Theorem 3.1. *The exponential function $z \mapsto e^z$ is an entire function.*

Proof. Well, we have

$$\frac{\partial f}{\partial x} = e^x(\cos y + i \sin y),$$

$$\frac{\partial f}{\partial y} = e^x(-\sin y + i \cos x) = i(e^x(\cos x + i \sin x)) = i\frac{\partial f}{\partial x}.$$

Thus $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ both exist, are continuous, and $\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$. So f is entire by Theorem 2.17! □

But is this a reasonable definition of e^z? To be honest, in the fullnesss of time we will check that this agrees with the expected power series definition. But for now we mention a different approach: first, if $z = x \in \mathbb{R}$ then certainly $e^z = e^x$ is the usual exponential function. Now here is a truly remarkable fact that we will prove towards the end of the course.

Theorem 3.2. *(Identity Theorem) Let $f, g : \mathbb{C} \to \mathbb{C}$ be two entire functions. Let*

$$S = \{z \in \mathbb{C} \mid f(z) = g(z)\}$$

be the set on which f and g agree. If S has an accumulation point, then $S = \mathbb{C}$: that is, $f = g$.

So suppose that we have two entire functions $f, g : \mathbb{C} \to \mathbb{C}$ such that for all $x \in \mathbb{R}$, we have

$$f(x) = e^x = g(x).$$

Then the set on which f and g agree contains the real axis $\mathbb{R}$, which has accumulation points (every point is an accumulation point). By the Identity Theorem, we conclude $f = g$. In other words, there is at most one entire function that restricts to the usual exponential function on the real axis. Therefore the weird definition we gave must be the correct one!

Proposition 3.3. *Let $z, z_1, z_2 \in \mathbb{C}$. Then:*
a) $e^{z_1}e^{z_2} = e^{z_1+z_2}$.
b) $\frac{1}{e^z} = e^{-z}$.
c) $e^{z+2\pi i} = e^z$.
d) $|e^z| = e^{\Re(z)}$.
e) $\frac{de^z}{dz} = e^z$.

Proof. a) If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ then

$$e^{z_1}e^{z_2} = e^{x_1}e^{iy_1}e^{x_2}e^{iy_2} = e^{x_1+x_2}e^{i(y_1+y_2)} = e^{z_1+z_2}.$$

b) If $z = x + iy$, then

$$\frac{1}{z} = \frac{1}{e^x}\frac{1}{e^{iy}} = e^{-x}e^{-iy} = e^{-z}.$$

c) Let $z = x + iy$. Then

$$e^{z+2\pi i} = e^x e^{(y+2\pi)i} = e^x(\cos(y + 2\pi) + i \sin(y + 2\pi)) = e^x(\cos x + i \sin y) = e^z.$$

d) Let $z = x + iy$. Then $|e^z| = |e^x e^{iy}| = |e^x||e^{iy}| = e^x \cdot 1 = e^x$.
e) In fact we did precisely this computation above. □

3.2. **The extended complex plane.** In calculus one extends the notion of limit in several ways:
(i) By defining $\lim_{x\to a} f(x) = \infty$, $\lim_{x\to a} f(x) = -\infty$.
(ii) By defining $\lim_{x\to\infty} f(x) = L$, $\lim_{x\to-\infty} f(x) = M$.

There are analogues of both of these concepts in complex analysis. However, whereas in $\mathbb{R}$ we distinguish between ∞ and $-\infty$, in $\mathbb{C}$ we just have one ∞. Here is a way to think about this: consider removing from $\mathbb{R}$ a large disk – i.e., closed interval. The complement is disconnected: it consists of a "positive" interval and a "negative" interval. However, if we remove a disk from $\mathbb{C}$ we still get a connected set.

Let $A \subset \mathbb{C}$, and let $a \in A$ be an accumulation point, and let $f : A \setminus \{a\} \to \mathbb{C}$ be a function. Then we write
$$\lim_{z\to a} f(z) = \infty$$
if: for all $M > 0$, there is $\delta > 0$ such that for all $z \in A \setminus \{a\}$, if $|z - a| < \delta$ then $|f(z)| > M$.

In other words, the limit is ∞ is we can make $f(z)$ arbitrarily large in magnitude by taking z close enough to a.

Example 3.4. *We have*
$$\lim_{z\to 0} \frac{1}{z} = \infty.$$
Indeed, if $M > 0$, *take* $\delta = \frac{1}{M}$. *Then if* $0 < |z - 0| = |z| < \delta = \frac{1}{M}$ *we have*
$$|f(z)| = |\frac{1}{z}| > \frac{1}{\delta} = M.$$
Similarly, for any $n \in \mathbb{Z}^+$ *we have*
$$\lim_{z\to 0} \frac{1}{z^n} = \infty.$$

Exercise 3.1. *Let* $P(z), Q(z)$ *be polynomials with* $Q(z) \neq 0$. *Let* $a \in \mathbb{C}$ *be such that* $Q(a) = 0$ *and* $P(a) \neq 0$. *Show:*
$$\lim_{z\to a} \frac{P(z)}{Q(z)} = \infty.$$

Let $A \subset \mathbb{C}$ be unbounded, and let $f : A \to \mathbb{C}$ be a function.
• For $L \in \mathbb{C}$, we say that
$$\lim_{z\to\infty} f(z) = L$$
if: for all $\epsilon > 0$, there is $M > 0$ such that for all $z \in A$, if $|z| \geq M$, then $|f(z) - L| < \epsilon$.

In other words, this means that we can make $f(z)$ arbitarily close to L by taking z to have large enough magnitude.

• We say that
$$\lim_{z\to\infty} f(z) = \infty$$
if for all $N > 0$ there is $M > 0$ such that for all $z \in A$, if $|z| \geq M$, then $|f(z)| \geq N$.

In other words, this means that we can make $f(z)$ arbitrarily large (in magnitude) by taking z arbitrarily large (in magnitude).

Exercise 3.2. *Let* $m, n \in \mathbb{Z}^+$ *and let*

$$P(z) = a_m z^n + \ldots + a_1 z + a_0,\ Q(z) = b_n z^n + \ldots + b_1 z + b_0$$

be polynomials with $a_m, b_n \neq 0$*. Let* $f(z) = \frac{P(z)}{Q(z)}$*.*
a) Suppose $m > n$*. Show:* $\lim_{z\to\infty} P(z) = \infty$*.*
b) Supose $m = n$*. Show:* $\lim_{z\to\infty} P(z) = \frac{a_m}{b_n}$*.*
c) Suppose $m < n$*. Show:* $\lim_{z\to\infty} P(z) = 0$*.*

We define the **extended complex plane** $\hat{\mathbb{C}}$ to be $\mathbb{C} \cup \infty$. Soon enough we will understand $\hat{\mathbb{C}}$ geometrically and explain why it can be viewed as the Riemann sphere. But for now we observe that every rational function can be viewed as a function from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$. We single out two examples:

• Suppose $R(z) = P(z)$ is a polynomial of positive degree. Then P is already an entire function $\mathbb{C} \to \mathbb{C}$. By the above exercise, we have $\lim_{z\to\infty} P(z) = \infty$, so it makes sense to define $P(\infty) = \infty$.

• Suppose $R(z) = \frac{1}{z}$. Then $f : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$. In our previous sense f was not defined at 0, but since $\lim_{z\to 0} R(z) = \infty$, in our extended sense we have $R(0) = \infty$. Moreover, we have $\lim_{z\to\infty} R(z) = 0$, so in our extended sense we have $R(\infty) = 0$. In this case, $R(z)$ gives a bijection from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$.

3.3. **Möbius transformations.** For $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$, we define the Möbius[7] transformation $f(z) = \frac{az+b}{cz+d}$. If $c = 0$ this function is a polynomial and hence already well understood, so in our calculations we will often assume that $c \neq 0$. This function is holomorphic on $\mathbb{C} \setminus \{\frac{-d}{c}\}$. Moreover, the condition $ad - bc$ ensures that the numerator is not zero at $\frac{-d}{c}$:

$$a(\frac{-d}{c}) + b = \frac{-ad + bc}{c} \neq 0.$$

By Exercise 3.2, this means that

$$\lim_{z\to\frac{-d}{c}} f(z) = \infty.$$

We claim that f gives a bijection from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$. Indeed, we can compute the inverse function explicitly: for $w \in \hat{\mathbb{C}}$, let

$$\frac{az+b}{cz+d} = w.$$

Then

$$(cz + d)w = az + b,$$

so

$$(a - cw)z = dw - b,\ z = \frac{dw - b}{-cw + a} =: g(w) \tag{10}$$

This computes the inverse explicitly and shows that it is another Möbius transformation. Moreover, in the inverse transformation the denominator is 0 precisely when $w = \frac{a}{c}$ which is precisely $f(\infty) = \lim_{z\to\infty} f(z)$, whereas

$$g(\infty) = \lim_{w\to\infty} g(w) = \frac{-d}{c},$$

so f and g are mutually inverse bijections from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$.

[7]August Ferdinand Möbius, 1790-1868, was a German mathematician and astronomer.

Exercise 3.3. *Let $a, b, c, d \in \mathbb{C}$. Show: the following are equivalent:*
(i) The function $f(z)\frac{az+b}{cz+d}$ is constant.
b) We have $ad - bc = 0$.

Is you matrix sense tingling? It should be. The condition $ad - bc \neq 0$ is clearly the condition that the matrix $\left[\begin{array}{cc} a & b \\ c & d \end{array}\right]$ be nonsingular. Moreover, the formula obtained for the inverse Möbius transformation to $\frac{az+b}{cz+d}$ is very similar to that of the inverse matrix to the matrix

$$M = \left[\begin{array}{cc} a & b \\ c & d \end{array}\right].$$

In fact, looking back at (10) we see that we have precisely the formula for the inverse matrix provided that $ad - bc = 1$. But in fact we can reduce to this case! First observe that if we scale the matrix M by $\lambda \in \mathbb{C}$ then the corresponding Möbius transformation is

$$\frac{\lambda az + \lambda b}{\lambda cz + \lambda d} = \frac{\lambda}{\lambda} \frac{az+b}{cz+d} = f(z),$$

so scaling a matrix does not change the Möbious transformation. On the other hand, if we scale by λ then the determinant becomes $(\lambda a)(\lambda d) - (\lambda b)(\lambda c) = \lambda^2(ad - bc)$, thus the determinant scales by λ^2. But now we can make use of the fact that every nonzero complex number has a square root, so there is $\lambda \in \mathbb{C}$ (in fact there are two – either will do!) such that $\lambda^2 = \frac{1}{ad-bc}$. Then if we scale by λ we get a matrix with determinant 1.

Notationally, we denote the set of all 2×2 matrices in $\mathbb{C}$ with nonzero determinant as $\mathrm{GL}_2(\mathbb{C})$ and the set of all 2×2 matrices in $\mathbb{C}$ with determinant 1 as $\mathrm{SL}_2(\mathbb{C})$. These are both groups under matrix multiplication, and $\mathrm{SL}_2(\mathbb{C}) \subset \mathrm{GL}_2(\mathbb{C})$.

Proposition 3.5. *For any $A \in \mathrm{GL}_2(\mathbb{C})$, let $f_A(z)$ be the corresponding Möbius transformation.*
a) We have $f_{A^{-1}}(z) = f_A(z)^{-1}$.
b) For $A, B \in \mathrm{GL}_2(\mathbb{C})$, we have $f_A(z)f_B(z) = f_{AB}(z)$.
c) For $A \in \mathrm{GL}_2(\mathbb{C})$, show that the following are equivalent:
(i) We have $f_A(z) = z$ for all $z \in \hat{\mathbb{C}}$.
(ii) We have that A is a scalar matrix.

Exercise 3.4. *Prove it!*

The next thing to observe is that for any Möbius transformation $f(z) = \frac{az+b}{cz+d}$, we have

$$f'(z) = \frac{(cz+d)a - (az+b)(c)}{(cz+d)^2} = \frac{ad-bc}{(cz+d)^2} \neq 0.$$

Therefore the Möbius transformations give conformal maps.

To further understand Möbius transformations it is helpful to write them as compositions of simpler Möbius transformations. Here are three especially simple types:
- Translation $f(z) = z + b$
- Dilation $f(z) = az$
- Inversion $f(z) = \frac{1}{z}$.

Proposition 3.6. *Let $f(z) = \frac{az+b}{cz+d}$ be a Möbius transformation.*
a) If $c = 0$, then $f(z) = \frac{a}{d}z + \frac{b}{d}$ is a dilation followed by a translation.

b) If $c \neq 0$, then

$$f(z) = \frac{bc-ad}{c^2}\frac{1}{z+\frac{d}{c}} + \frac{a}{c}$$

is a translation followed by an inversion followed by a dilation followed by a translation.

Exercise 3.5. *Prove it!*

We want to use this to study a mapping property of Möbius transformations. We claim that if $A \subset \mathbb{C}$ is either a circle or a line and $f(z)$ is a Möbius transformation, then $f(A)$ is again either a circle or a line. By Proposition 3.6 it is enough to show this for translations, dilations and inversion separately. Clearly, if we translate a line then we get a line and if we translate a circle then we get a circle. Since a dilation is the composition of a rotation and a scaling, again if we dilate a line then we get a line and if we dilate a circle then we get a circle.

Inversions are more interesting.
• Since inversion acts geometrically as conjugating and then rescaling the length as the reciprocal, if $\ell = \{y = mx\}$ is any line *through the origin*, then inversion carries ℓ to another line, namely $\{y = -mx\}$.
• Inversion takes the unit circle to itself.
• However, let ℓ be the line $y = 1$. Since every point of ℓ lies on or outside the unit circle, its distance from the origin is at least one. Therefore, after inversion, the distance of every point from the origin is at most one. So it cannot be that inversion takes ℓ to another line! A picture suggests that inversion takes ℓ to the circle of radius $\frac{1}{2}$ centered at $\frac{-i}{2}$. So we calculate:

$$|f(x+i) - \frac{-i}{2}|^2 = |\frac{1}{x+i} + \frac{i}{2}|^2 = |\frac{2+(x+i)i}{2x+2i}|^2 = |\frac{1+xi}{2x+2i}|^2 = \frac{x^2+1}{4x^2+4} = \frac{1}{4}.$$

Correct! Notice that inversion does not take any point in $\mathbb{C}$ to 0 but it takes ∞ to 0.
• Because inversion is its own inverse, inversion takes the circle of radius $\frac{1}{2}$ centered at $\frac{-i}{2}$ to the line $y = 1$. Moreover, if C is any circle passing through 0, then C contains points arbitrarily close to 0, so after inversion contains points arbitrarily far away from 0, so $f(C)$ cannot be a circle.

Theorem 3.7. *Let $A \subset \mathbb{C}$ be either a line or a circle, and let $f(z)$ be a Möbius transformation. Then $f(A)$ is contained in a line or contained in a circle.*

Proof. See [BMPS, p. 35]. □

It is a basic fact of plane geometry that through any three noncollinear points there is a unique circle. (The center of the circle will be equidistant from the three points. So to find it we consider the three lines that are the perpendicular bisectors of the three pairs of points. These three points intersect in a unique point, as one can show e.g. by elementary vector calculations.) If we have three collinear points then there is no circle passing through them. However, given two points $P_1 \neq P_2 \in \mathbb{C}$, consider the line ℓ passing through them. Since ℓ contains points that are arbitrarily far from the origin, it makes some sense to say that ℓ passes arbitrarily close to the point ∞. We regard the line ℓ together with the point at ∞ as a circle in the extended complex plane $\hat{\mathbb{C}}$. (When we introduce stereographic projection later, we can make more honest geometric sense out of this.) In this sense then, for any three points P_1, P_2, P_3 in $\hat{\mathbb{C}}$ there is a unique circle passing through them. This circle contains ∞ iff either one of the points is ∞ or the three points are collinear.

Theorem 3.8. *Given any two triples (z_1, z_2, z_3) and (w_1, w_2, w_3) of distinct points in $\hat{\mathbb{C}}$, there is a* unique *Möbius transformation $f(z)$ such that*

$$f(z_1) = w_1, \ f(z_2) = w_2, \ f(z_3) = w_3.$$

Proof. **Step 1:** Suppose that for any z_1, z_2, z_3 there is a Möbius transformation $f_{(z_1,z_2,z_3)}$ such that

$$f(z_1) = 0,\ f(z_2) = 1,\ f(z_3) = \infty.$$

Then

$$f := f^{-1}_{(w_1,w_2,w_3)} \circ f_{(z_1,z_2,z_3)}$$

is a Möbius transformation that maps $z_1 \mapsto w_1$, $z_2 \mapsto w_2$, $z_3 \mapsto w_3$: e.g. we have

$$f(z_1) = f^{-1}_{(w_1,w_2,w_3)}(f_{(z_1,z_2,z_3)}(0)) = f^{-1}_{(w_1,w_2,w_3)}(0) = w_1,$$

and similarly for z_2 and z_3.
Step 2: We define

$$f_{(z_1,z_2,z_3)} := \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}.$$

This works: $f(z_1) = 0$, $f(z_2) = 1$ and $f(z_3) = \infty$. (When one of z_1, z_2, z_3 is ∞, this results in an instance of ∞ in both the numerator and the denominator, and our convention is to cancel these, e.g. $f_{(z_1,z_2,\infty)} = \frac{z-z_1}{z_2-z_1}$.)
Step 3: Suppose that $f(z) = \frac{az+b}{cz+d}$ is a Möbius transformation such that

$$f(0) = 0,\ f(1) = 1, f(\infty) = \infty.$$

Then f is the identity function – i.e., $b = c = 0$ and $a = d$.
Indeed, we calculate

$$0 = f(0) = \frac{b}{d},\ \text{so } b = 0.$$

$$\infty = f(\infty) = \frac{a}{c},\ \text{so } c = 0.$$

$$1 = f(1) = \frac{a(1)+b}{c(1)+d} = \frac{a}{d},\ \text{so } a = d.$$

Step 4: Let f be a Möbius transformation, and let $z_1, z_2, z_3 \in \hat{\mathbb{C}}$ be distinct points. If $f(z_i) = z_i$ for $i \in \{1, 2, 3\}$, then f is the identity function.

Indeed, by Step 2 there is a Möbius transformation γ such that $\gamma(z_1) = 0$, $\gamma(z_2) = 1$, $\gamma(z_3) = \infty$. Then the Möbius transformation $\gamma f \gamma^{-1}$ maps 0 to 0, 1 to 1 and ∞ to ∞, so by Step 3 we have

$$\gamma f \gamma^{-1} = 1.$$

Multiplying on the left by γ^{-1} and on the right by γ, we get

$$f = \gamma^{-1}\gamma = 1.$$

Step 5: Let f, g be Möbius transformations, and let $z_1, z_2, z_3 \in \hat{\mathbb{C}}$ be three distinct points such that $f(z_i) = g(z_i)$ for all $i \in \{1, 2, 3\}$. Then $f = g$.

Indeed, $g^{-1} \circ f$ fixes z_1, z_2 and z_3, so by Step 5 we have $g^{-1} \circ f = 1$, hence $f = g$. ☐

In the course of the above proof we introduced a quantity that has further importance. Namely, let z, z_1, z_2, z_3 be distinct points of $\hat{\mathbb{C}}$. We define their **cross ratio**

$$[z, z_1, z_2, z_3] := f_{(z_1,z_2,z_3)}(z) := \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} \in \mathbb{C}.$$

Proposition 3.9. *Möbius transformations preserve cross-ratios. That is, for distinct points* $z, z_1, z_2, z_3 \in \hat{\mathbb{C}}$ *and a Möbius transformation* f*, we have*

$$[f(z), f(z_1), f(z_2), f(z_3)] = [z, z_1, z_2, z_3].$$

Proof. Let us first regard z as a variable. $[z, z_1, z_2, z_3]$ and $[f(z), f(z_1), f(z_2), f(z_3)]$ are, as functions of z, Möbius transformations (the latter is a composition of two Möbius tranformations, hence a Möbius transformation), say S and T. We calculate:

$$T(z_1) = \frac{(f(z_1) - f(z_1))(f(z_2) - f(z_3))}{(f(z_1) - f(z_3))(f(z_2) - f(z_1))} = 0 = S(z_1),$$

$$T(z_2) = \frac{(f(z_2) - f(z_1))(f(z_2) - f(z_3))}{(f(z_2) - f(z_3))(f(z_2) - f(z_1))} = 1 = S(z_2),$$

$$T(z_3) = \frac{(f(z_3) - f(z_1))(f(z_2) - f(z_3))}{(f(z_3) - f(z_3))(f(z_2) - f(z_1)} = \infty = S(z_3).$$

By Theorem 3.8 we conclude $S = T$. Now "evaluating the variable z at $z \in \hat{\mathbb{C}}$" gives the result. □

Theorem 3.10. *For $z, z_1, z_2, z_3 \in \hat{\mathbb{C}}$, the following are equivalent:*
(i) All four points z, z_1, z_2, z_3 lie on a circle in $\hat{\mathbb{C}}$.
(ii) The cross-ratio $[z, z_1, z_3, z_3]$ lies in $\mathbb{R}$.

Proof. See [Co, Prop. 3.10]. □

3.4. The Riemann sphere.

Unfortunately this section is currently blank.

4. Contour integration

We will now pursue the theory of integration in the complex plane. It is a close analogue of the **line integral** of multi-variable calculus, but with some particular features of its own.

We will begin by reviewing the notion of a line integral in the plane. Let $U \subset \mathbb{C}$ be an open subset, let $\gamma : [a, b] \to U$ be a smooth path, and let $F : U \to \mathbb{R}^2$ be a smooth function. In other words, $F(x, y) = (P(x, y), Q(x, y))$ is a **vector field** defined on U, and the line integral $\int_\gamma F$ captures the work done as a particle moves through the vector field F along the path γ. This comes down to integrating the inner product of the vector field with the tangent vector as we move along the path:

$$\int_\gamma F = \int_a^b F(\gamma(t)) \cdot \gamma'(t) dt \in \mathbb{R}.$$

Let us briefly recall some aspects of line integration:

• A parameterized curve determines an **oriented path** in the plane. The line integral depends only on the oriented path, not the particular parameterization. This comes down to a chain rule calculation. However, if we traverse the path with the opposite orientation, then the effect on the line integral is multiplication by -1.
• We can define the line integral along a piecewise smooth path simply as the sum of line integrals along the smooth paths that comprise the piecewise smooth path.
• A vector field $F : U \to \mathbb{R}^2$ is called **conservative** if it satisfies any of the following equivalent properties:
(i) Given two points $P, Q \in U$, for all oriented paths γ_1 and γ_2 from P to Q, we have

$$\int_{\gamma_1} F = \int_{\gamma_2} F.$$

(ii) For all closed paths γ – i.e., $\gamma(a) = \gamma(b)$ – we have $\int_\gamma F = 0$.
(ii′) For all *simple* closed paths γ we have $\int_\gamma F = 0$.
(iii) There is a function $f : U \to \mathbb{R}$ such that $F = \mathrm{grad}(f)$: that is,

$$\frac{\partial f}{\partial x} = P, \ \frac{\partial f}{\partial y} = Q.$$

Also consider condition (iv) A vector field $F = (P, Q)$ is **irrotational** if $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial Y} \equiv 0$ on U.

Then it turns out that all conservative vector fields are irrotational. This is easy to see: if $P = \frac{\partial f}{\partial x}$ and $Q = \frac{\partial f}{\partial y}$, then

$$\frac{\partial Q}{\partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial P}{\partial Y}.$$

Depending upon U it may or may not be the case that all irrotational vector fields are conservative.

Example 4.1. *Let $D = \mathbb{R}^2 \setminus \{0\}$, and let $F : D \to \mathbb{R}^2$ by*

$$F(x, y) = (P(x, y), Q(x, y)) = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right).$$

Then F is irrotational:

$$\frac{\partial Q}{\partial x} = \frac{y^2 - x^2}{(x^2 + y^2)^2} = \frac{\partial P}{\partial y}.$$

On the other hand, let $\gamma : [0, 2\pi] \to \mathbb{R}^2$ by $t \mapsto (\cos t, \sin t)$ parameterize the unit circle. Then

$$\int_\gamma F = \int_0^{2\pi} \left(\frac{-\sin t}{\cos^2 t + \sin^2 t}, \frac{\cos t}{\cos^2 t + \sin^2 t} \right) \cdot (-\sin t, \cos t) dt = \int_0^{2\pi} \sin^2 t + \cos^2 t dt = \int_0^{2\pi} 1 dt = 2\pi.$$

It follows that there is no function $f : D \to \mathbb{R}$ such that $F = \mathrm{grad}(f)$. However there almost *is such a function. Namely, let us take the gradient of θ. If we write $\theta = \arctan(y/x)$, then formally we have*

$$\mathrm{grad}(\theta) = \left(\frac{\partial \arctan(y/x)}{\partial x}, \frac{\partial \arctan(y/x)}{\partial y} \right) = \left(\frac{-y/x^2}{1 + (y/x)^2}, \frac{1/x}{1 + (y/x)^2} \right) = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right) = F.$$

The problem is that $\theta = \arctan(y/x)$ is only valid when $x > 0$. In fact this shows something startling: the argument cannot be extended to a single-valued *differentiable function on all of D. In fact, not even a continuous function: the idea is that as we wind around a unit circle the argument continuously increments so that when we get back where we started the angle is larger by a value of 2π. Notice that the integral that we evaluated was 2π – not a concidence!*

Let us say that a vector field $F = (P, Q)$ defined on $D \subset \mathbb{C}$ is **smooth** if all the partial derivatives $\frac{\partial P}{\partial x}, \frac{\partial P}{\partial Y}, \frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial y}$ exist and are continuous.

Theorem 4.2. *(Green[8]'s Theorem) Let $D \subset \mathbb{C}$ be a bounded domain with boundary C, and let $F = (P, Q)$ be a smooth vector field defined on D. Then we have*

$$\int \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = \int_C F. \tag{11}$$

[8] George Green, 1793-1841, was an English mathematical physicist. Many of the basic concepts of vector analysis and their applications to electro-magnetism are due to him.

We call a domain $D \subset \mathbb{C}$ **simply connected** if every simple closed curve in D is the boundary of a simple closed curve γ with image lying in D.

Examples of simply connected domains include: $\mathbb{C}$ itself, any open disk in $\mathbb{C}$, and any domain that is convex – for any two points $P, Q \in D$, the line segment joining P to Q lies entirely in D. A more interesting example is the interior of any simple closed curve in the plane. (To be honest, this is actually a theorem of topology: the Jordan-Schoenflies Theorem.)

An example of a domain that is not simply connected is $\mathbb{C} \setminus \{0\}$. The unit circle is not the boundary of any domain that excludes 0.

Theorem 4.3. *Let $D \subset \mathbb{C}$ be a simply connected domain. Then every irrotational vector field $F : U \to \mathbb{R}^2$ is conservative.*

Proof. Let γ be a simple closed path with image curve C. Since D is simply connected, C is the boundary of a subregion $D' \subset D$. By Green's Theorem, we have

$$\int_C F = \int \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = 0. \qquad \square$$

Now we turn to the notion of a complex contour integral. Interestingly, the data is essentially identical: in place of a vector field $F : D \to \mathbb{R}^2$ we give ourselves a function $f : D \to \mathbb{C}$ and a path $\gamma : [a, b] \to D$. Notice that $\mathbb{C}$ is just $\mathbb{R}^2$ but with an extra multiplication structure. So whereas before we integrated the scalar product $F(\gamma(t))$ with $\gamma'(t)$, now $F(\gamma(t))$ and $\gamma'(t)$ are both complex numbers, so we multiply them:

$$\int_\gamma f(z)dz := \int_a^b f(\gamma(z))\gamma'(z)dz.$$

We do need one word about the the final integral: we are integrating a $\mathbb{C}$-valued function. But as usual, we should view a $\mathbb{C}$-valued function in terms of its real and imaginary parts, and we integrate each separately.

Example 4.4. *Let $f : \mathbb{C} \to \mathbb{C}$ be given by $f(z) = \overline{z}^2$. (Note: not a holomorphic function!) We will compute the line integral of f over several different paths each starting at 0 and ending at $1 + i$.*
a) Let γ be the straight line segment from 0 to $1 + i$. One parameterization of this path is $\gamma : [0, 1] \to \mathbb{C}$, $\gamma(t) = (1 + i)t$. Then $\gamma'(t) = 1 + i$, and we have

$$\int_\gamma f = \int_0^1 f(\gamma(t))\gamma'(t)dt = \int_0^1 (\overline{(1 + i)t})^2(1 + i)dt$$

$$= (\overline{1 + i})^2(1 + i) \int_0^1 t^2 dt = \frac{1}{3}(1 - i)^2(1 + i) = \frac{1}{3}|1 + i|^2(1 - i) = \frac{2}{3}(1 - i).$$

b) Let γ be the arc of the parabola $y = x^2$ from 0 to $1 + i$. So the natural paramterization is $t \mapsto (t, t^2) = t + it^2$ for $t \in [0, 1]$, i.e., $\gamma(t) = t + it^2$. Then

$$\int_\gamma f = \int_0^1 \gamma(t)\gamma'(t)dt = \int_0^1 (\overline{t + it^2})^2(1 + 2it)dt$$

$$= \int_0^1 (t - it^2)^2(1 + 2it)dt = \int_0^1 (t^2 + 3t^4 - 2it^5)dt = \frac{1}{3} + \frac{3}{5} - \frac{2i}{6} = \frac{14}{15} - \frac{i}{3}.$$

One cannot help but notice that we did not get the same answer both times. It appears that the – ahem, nonholomorphic!! – function $f(z) = \overline{z}^2$ fails to satisfy the contour integral analogue of being a conservative vector field.

One difference between calculus and analysis is that in calculus we try to compute things exactly (if possible, which by the way it most often is not!) and in analysis we come up with useful estimates. Despite what you learn in freshman calculus, if $f : [a,b] \to \mathbb{R}$ is continuous (or infinitely differentiable, or whatever) then it is very unlikely that we can evaluate $\int_a^b f$ exactly. Both in theory and in applications, methods of approximation of $\int_a^b f$ can be more useful. The simplest possible approximation is the following: if $||f||$ denotes the maximum of $|f(x)|$ on $[a,b]$, then

$$|\int_a^b f| \leq ||f||(b-a).$$

The following is the analogous (and closely related) result for contour integrals.

Proposition 4.5. *("ML Inequality") Let $f : D \to \mathbb{C}$ be a continuous function and let $\gamma : [a,b] \to D$ be a piecewise smooth path. Let M be the maximum value of $|f(\gamma(t))|$ for $t \in [a,b]$ and let L be the length of the path γ – that is,*

$$L := \int_a^b |\gamma'(t)|dt.$$

Then

$$|\int_\gamma f| \leq ML.$$

Proof. We may write $\int_\gamma f = Re^{i\theta}$, so $R = |\int_\gamma f|$. Then

$$R = e^{-i\theta}\int_\gamma f = \Re(e^{-i\theta}\int_a^b f(\gamma(t))\gamma'(t)dt) =$$

$$\int_a^b \Re(f(\gamma(t))e^{-i\theta}\gamma'(t)dt) \leq \int_a^b |f(\gamma(t))e^{-i\theta}\gamma'(t)dt|$$

$$= \int_a^b |f(\gamma(t))||\gamma'(t)|dt \leq M\int_a^b |\gamma'(t)|dt = ML. \qquad \square$$

In one variable calculus our best tool for evaluating integrals exactly is by the Fundamental Theorem of Calculus, one half of which states that if F is an antiderivative of f, then

$$\int_a^b f = F(b) - F(a).$$

There is an analogue for complex contour integrals; in fact it is quite straightforward but just as useful. If $D \subset \mathbb{C}$ is a region and $f : D \to \mathbb{C}$ is a function, we say that $F : D \to \mathbb{C}$ is an **antiderivative** of f on D if (what else?!?) $F'(z) = f(z)$ for all $z \in D$. Because we have the Zero Velocity Theorem for complex functions, it is still true that any two antiderivatives of a given function differ by a constant. (It is much less clear which functions $f : D \to \mathbb{C}$ have antiderivatives. In fact it is not true that every continuous function has an antiderivative...which sounds bad but is actually good. Wait a while to see how the story plays out.)

Theorem 4.6. *Let $D \subset \mathbb{C}$ be a domain. Suppose $f : D \to \mathbb{C}$ is continuous and that $F : D \to \mathbb{C}$ is an antiderivative of F.*
a) If $\gamma : [a,b] \to D$ is any piecewise smooth path, then we have

$$\int_\gamma f = F(\gamma(b)) - F(\gamma(a)).$$

b) In particular, if γ is any closed path – i.e., $\gamma(b) = \gamma(a)$, then $\int_\gamma f = 0$.

Proof. a) Since by the Chain Rule for paths, we have $\frac{d}{dt}F(\gamma(t)) = f(\gamma(t))\gamma'(t)$, this is immediate from the Fundamental Theorem of Calculus that we just recalled:

$$\int_\gamma f = \int_a^b f(\gamma(t))\gamma'(t)dt = \int_a^b \frac{d}{dt}F(\gamma(t)) = F(\gamma(b)) - F(\gamma(a)).$$

b) This follows immediately. □

Notice that Theorem 4.6 is the analogoue for complex contour integrals of the aforementioned fact for line integrals that a gradient vector field is conservative (and the proof is closely related).

Example 4.7. *Let $n \in \mathbb{Z}$, and let $f_n : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ given by $f(z) = z^n$, and let $\gamma : [0, 2\pi] \to \mathbb{C}$ given by $\gamma(t) = e^{iz}$. First suppose that $n \neq -1$. Then $F(z) = \frac{z^{n+1}}{n+1}$ is an antiderivative of f, so*

$$\int_\gamma f = F(\gamma(2\pi)) - F(\gamma(0)) = F(1) - F(1) = 0.$$

On the other hand, if $n = -1$, then

$$\int_\gamma f = \int_0^{2\pi} (e^{-iz})\frac{de^{iz}}{dz}dz = \int_0^{2\pi} e^{-iz}ie^{iz}dz = i\int_0^{2\pi} 1dz = 2\pi i.$$

Note that this proves *that $f(z) = \frac{1}{z}$ does not have an antiderivative on $\mathbb{C}\setminus\{0\}$. It follows easily that there is no function $L : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ such that $e^{L(z)} = z$ for all $z \in \mathbb{C} \setminus \{0\}$. Indeed, differentiating both sides gives*

$$1 = e^{L(z)}L'(z),$$

so

$$L'(z) = \frac{1}{e^{L(z)}} = \frac{1}{z}.$$

Thus – as in calculus – an inverse function to the exponential function is necessarily an antiderivative of $\frac{1}{z}$. This shows that logarithms really don't work as we may naively want to. However, later we will show that if D is simply connected, then every holomorphic function $f : D \to \mathbb{C}$ has an antiderivative, meaning that we can define a logarithm by suitably restricting the domain.

Now we give the converse to Theorem 4.6, which is the contour integral analogue of the fact that conservative vector fields are gradient fields.

Theorem 4.8. *Let $D \subset \mathbb{C}$ be a domain, let $z_0 \in D$, and let $f : D \to \mathbb{C}$ be a continuous function. Suppose that $\int_\gamma f = 0$ for any simple closed polygonal path $\gamma : [a, b] \to D$. For any $z \in D$, let γ_z be any polygonal path connecting z_0 to z. Then*

$$F(z) := \int_{\gamma_z} f$$

is an antiderivative for f on D.

Proof.
Step 0: Because D is a domain it is xy-connected, so there is a polygonal path from z_0 to z.
Step 1: Our first order of business is to show that $F(z)$ is well-defined independent of the choice of polygonal path. But this is easy: suppose we had two such paths γ_1 and γ_2. Then the path γ_3 obtained by doing γ_1 followed by γ_2 with the orientation reversed is a polygonal path going from z_0 to z and back, i.e., is a *closed* polygonal path in D. It follows that

$$\int_{\gamma_3} f = \int_{\gamma_1} f - \int_{\gamma_2} f,$$

so it is enough to show that $\int_{\gamma_3} f = 0$. For this we can *almost* apply our hypothesis, with one minor technicality: the closed path γ in the statement is assumed to be *simple*, i.e., not have any self-intersections. But fortunately we are dealing with polygonal paths. Given a closed polygonal path, we first remove instances of "backtracking" – i.e., if we traverse a line segment in one direction and the same line segment in the opposite direction, the contributions to the integral cancel out, and if we remove this pair of line segments we still have a closed path. After having done that, we get a closed polygonal path that is a finite union of simple closed polygonal paths. (Here you must draw a picture. Sorry for not including one in these notes.) And then we can apply our hypothesis to each of the simple closed paths to get that the integral is zero on each, hence the overall integral is 0. So indeed $F(z)$ is well-defined.
Step 2: Let $z \in D$ and let $h \in \mathbb{C}$ be small enough such that $z+h$ and the line segment from z to $z+h$ are both in D. (Since by definition of domain, D contains a small open disk centered at z, this is certainly permissible.) Then

$$F(z+h) - F(z) = \int_{\gamma_{z+h}} f - \int_{\gamma_z} f = \int_{\gamma} f$$

where γ is any polygonal path from z to $z+h$. The constant function 1 has antiderivative z, so applying Theorem 4.6 we know that $\int_\gamma 1 = h$. So

$$\frac{F(z+h)-F(z)}{h} - f(z) = \frac{1}{h}\int_\gamma f(w)dw - \frac{f(z)}{h}\int_\gamma dw = \frac{1}{h}\int_\gamma (f(w)-f(z))dw.$$

Now let λ be the straight line path from z to $z+h$, which as above, we have arranged to be entirely contained in D. Using the argument of Step 1, we get

$$\frac{1}{h}\int_\gamma (f(w)-f(z))dz = \frac{1}{h}\int_\lambda (f(w)-f(z))dw.$$

Finally we are in a position to show that the last expression approaches 0 as $h \to 0$, which will complete the proof. Since f is continuous at z, for each $\epsilon > 0$ there is $\delta > 0$ such that $|w - z| < \delta \implies |f(w) - f(z)| < \epsilon$. Thus if h is sufficiently small as above and also $|h| < \delta$, the ML Inequality gives

$$|\frac{1}{h}\int_\lambda f(w)-f(z))dw| \leq \frac{1}{|h|}\max_{w\in\lambda}|f(w)-f(z)|\,\text{length}(\lambda)| = \max_{w\in\lambda}|f(w)-f(z)| < \epsilon.$$

(Here we have used that the length of the straight line path from z to $z+h$ is $|h|$!) $\square$

To sum up: for a continuous function $f : D \to \mathbb{C}$, the following are equivalent:
(i) f has an antiderivative F on D.
(ii) For $a, b \in D$ and any two piecewise smooth paths γ_1, γ_2 from a to b, we have $\int_{\gamma_1} f = \int_{\gamma_2} f$.
(iii) For any closed piecewise smooth path $\gamma : [a,b] \to \mathbb{C}$, we have $\int_\gamma f = 0$.
(iv) For any simple closed polygonal path $\gamma : [a,b] \to \mathbb{C}$, we have $\int_\gamma f = 0$.

Indeed, Theorem 4.6 gave us (i) $\implies$ (ii). (ii) and (iii) are equivalent using the trick of writing a closed path as a path from a to b followed by a path from b to a and conversely. (iii) $\implies$ (iv) is immediate, since polygonal paths are piecewise smooth, and (iv) $\implies$ (i) is Theorem 4.8. Notice that our use of polygonal paths is because it was easy to reduce a closed polygonal path to a simple closed polygonal path. In general if we take two smooth curves γ_1 and γ_2 then γ_1 followed by γ_2 in the opposite direction could have much more complicated self-intersection – imagine for instance γ_1 to be a portion of the x-axis and γ_2 to be the graph of $\sin(1/x)$. So this was the easy way around. Finally, note of course that this is a perfect analogue of our equivalent conditions for conservative vector fields.

Our next order of business is to show that if D is simply connected, then any holomorphic function satisfies all of these equivalent conditions. Consulting what we said above, we suspect that Green's Theorem should be helpful here. And indeed it is. The following theorem and proof is one of the highlights of the course.

Theorem 4.9. *(Cauchy's Integral Theorem, v.1)*
Let $D \subset \mathbb{C}$ be a simply connected domain, and let $f : D \to \mathbb{C}$ be a holomorphic function such that $f' : D \to \mathbb{C}$ is continuous. Then f for any piecewise smooth closed path γ in D, we have $\int_\gamma f = 0$.

Proof. By the above discussion it is sufficient to show that $\int_\gamma f = 0$ for any *simple* piecewise smooth closed path $\gamma : [a, b] \to D$. (We could also assume that the path is polygonal, but there is no need to do so.) Write $f(z) = u(z) + iv(z)$ as usual and $\gamma(t) = x(t) + iy(t)$. We define two vector fields $F_1, F_2 : D \to \mathbb{R}^2$ by

$$F_1(x, y) = (u(x, y), -v(x, y)),\ F_2(x, y) = (v(x, y), u(x, y)).$$

Then we have

$$\int_\gamma f = \int_a^b (u(\gamma(t)) + iv(\gamma(t))(x'(t) + iy'(t)dt)$$

$$= \int_a^b u(\gamma(t))x'(t) - v(\gamma(t))y'(t)dt + i\int_a^b (v(\gamma(t))x'(t) + u(\gamma(t))y'(t))dt$$

$$= \int_a^b F_1(\gamma(t)) \cdot \gamma'(t)dt + i\int_a^b F_2(\gamma(t)) \cdot \gamma'(t)dt = \int_\gamma F_1 + i\int_\gamma F_2.$$

Thus it transpires that our complex contour integral is not just analogous to a line integral; we can actually evaluate it in terms of a pair of line integrals. Since f is holomorphic, u and v satisfy the Cauchy-Riemann equations:

$$f'(z) = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial f}{\partial y} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}.$$

A complex valued function is continuous iff its real and imaginary parts are both continuous, so because we've assumed that f' is continuous, this implies that $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x}$ are continuous. Since $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, we conclude that F_1 and F_2 are smooth vector fields. Moreover, since D is simply connected, the simple closed curve γ is the boundary of a subregion $D' \subset D$, so we may apply Green's Theorem:

$$\int_\gamma F_1 = \int\int_{D'} \left(\frac{d(-v)}{dx} - \frac{du}{dy}\right) dxdy = 0,$$

$$\int_\gamma F_2 = \int\int_{D'} \left(\frac{du}{dx} - \frac{dv}{dy}\right) dxdy = 0,$$

so

$$\int_\gamma f = \int_\gamma F_1 + i\int_\gamma F_2 = 0 + i \cdot 0 = 0. \qquad \square$$

In particular, if γ is a simple closed path in $\mathbb{C}$ and f is holomorphic with a continuous derivative on an open set containing γ and its interior, then $\int_\gamma f = 0$. In this latter form the result was stated by A.-L. Cauchy in 1825. (A rudimentary version was presented by him to the French Academy of Sciences in 1814, when Cauchy was 24.) The proof that we have given here was not Cauchy's original proof but is a simplification dating from 1846...also by Cauchy.

4.1. **Cauchy-Goursat.** From a theoretical perspective there is a small fly in the ointment: do we really need the hypothesis that f' is continuous on D? The answer is that we needed it in order to apply Green's Theorem: in fact there are counterexamples to Green's Theorem if we only assume that our vector field $F = (P, Q)$ has all four partial derivatives. But this is a counterexample to this particular proof of Cauchy's Theorem, not the theorem itself. In fact Edouard Goursat[9] showed in 1884 that this hypothesis could be removed.

Theorem 4.10. *(Cauchy-Goursat Theorem) Let $D \subset \mathbb{C}$ be a simply connected domain, and let $f : D \to \mathbb{C}$ be a holomorphic function. Then f for any piecewise smooth closed path γ in D, we have $\int_\gamma f = 0$.*

Notice that in light of Theorems 4.6 and 4.8, it follows from Theorem 4.10 that every holomorphic function on a simply connected domain has an antidervative. (Again, recall that we know that "simply connected" cannot be omitted here, because of the holomorphic function $\frac{1}{z}$ on $\mathbb{C} \setminus \{0\}$.

We will prove Theorem 4.10 in two steps. First:

Theorem 4.11. *(Goursat's Lemma) Let $f : U \to \mathbb{C}$ be holomorphic, and let T be a triangle such that T and its interior are contained in U. Then $\int_T f = 0$.*

Proof. We give T the positive orientation. We shall repeatedly "bisect" T into four congruent triangles. We do this by drawing the three line segments between the pairs of midpoints of the three sides of T. It is easy to see that this gives four triangles each congruent to each other and similar to T, but with sides half as long. We call these triangles $T_1^1, T_2^1, T_3^1, T_4^1$. Next note that there is exactly one way to give positive orientations to all four smaller triangles that is consistent with the given positive orientation on T. When we do this, each of the three interior sides belongs to two of the smaller triangles, but the two orientations of the sides are in opposite directions. It follows that when we do line integrals around each of the four smaller triangles, the interior sides cancel out and we are left with the line integral around T:

$$\int_T f = \int_{T_1^1} f + \int_{T_2^1} f + \int_{T_3^1} f + \int_{T_4^1} f.$$

Now comes a simple but key observation: for at least one $1 \leq j \leq 4$, we must have

$$|\int_{T_j^1} f| \geq \frac{1}{4} |\int_T f|.$$

Indeed, if not, then $|\int_{T_j^1} f| < \frac{1}{4} |\int_T f|$ for all $1 \leq j \leq 4$, and then summing gives

$$|\int_T f| = |\int_{T_1^1} f + \int_{T_2^1} f + \int_{T_3^1} f + \int_{T_4^1} f| \leq \sum_{j=1}^4 |\int_{T_j^1} f| < 4\frac{1}{4} |\int_T f| = |\int_T f|,$$

which is absurd. We choose such a triangle and rename it T^1. Now we repeat the entire argument with T^1 in place of T, getting a subriangle T^2 with half the side lengths of T^1 and such that

$$\int_{T^2} f| \geq \frac{1}{4} |\int_{T^1} f| \geq 4^{-2} |\int_T f|.$$

Proceeding in this way, we get a nested sequence of triangles

$$T \supset T^1 \supset T_2 \supset \ldots \supset T^n$$

[9] Édouard Jean-Baptiste Goursat, 1858-1936, was a French mathematician. He worked primarily on firming up the "rigor" in mathematical analysis, and his *Cours d'analyse mathématique* was one of the first analysis textbooks that holds up to modern scrutiny.

each of twice the side length of the next and such that

$$|\int T^n f| \geq 4^{-n} |\int_T f|.$$

Let A_n denote T^n together with its interior. Choose a point $z_n \in A^n$. Let d_0 be the diameter of T, so the diameter d_n of A^n is $2^{-n}d_0$. Also the perimeter p_n of T^n is $2^{-n}p_0$, where p_0 is the perimeter of T. Since $d_n \to 0$ it follows that $\{z_n\}$ is a Cauchy sequence, and thus converges to a point a. For each fixed n we have $z_m \in A^n$ for all $m \geq n$ and thus a is the limit of a sequence of points lying inside the closed set A^n, so also $a \in A^n$. Since f is holomorphic at a, we may write

$$f(z) = f(a) + f'(a)(z-a) + \psi(z)(z-a)$$

for a function $\psi(z)$ with $\lim_{z\to a} \psi(z) = 0$. (Thus by defining $\psi(a) = 0$, ψ becomes a continuous function on a small disk around a with $\psi(0) = 0$.) Clearly linear functions have antiderivatives so integrate to 0 around closed paths, and thus we get

$$\int_{T^n} f(z) = \int_{T^n} \psi(z)(z-a).$$

Let ϵ_n be the maximum of $|\psi(z)|$ on T^n, so $\lim_{n\to\infty} \epsilon_n = 0$. Since for $z \in T^n$, $|z-a| \leq d_n = 2^{-n}d_0$, Then by the ML-Inequality we have

$$|\int_T f(z)| \leq 4^n |\int_{T^n} f(z)| = 4^n |\int_{T^n} \psi(z)(z-a)| \leq 4^n \frac{d_0 p_0}{4^n} \epsilon_n = d_0 p_0 \epsilon_n.$$

Since $\epsilon_n \to 0$, this shows that $\int_T f(z) = 0$, completing the proof. $\square$

The second step is the following:

Lemma 4.12. *Every simple closed polygon P can be triangulated.*

Proof. Let D be the closed, bounded planar set consisting of P together with its interior. We go by induction on the number of vertices n of P. The base case, $n = 3$, is that of a triangle: OK! So assume $n \geq 4$ and that every simple closed polygon with $3 \leq m \leq n-1$ vertices can be triangulated.

Certainly we can rotate the polygon if necessary without affecting the conclusion. After rotating the polygon, we may assume that it has a unique leftmost vertex, say v. (The only other possibility is that the polygon contains a vertical segment, but since there are only finitely many slopes of line segments of the sides of the polygon, a small rotation fixes this.) Let u and w be the two vertices adjacent to v, and consider the line segment $\overline{uw}$. If this line segment lies inside P, then we're good. More precisely, D is the union of the triangle uvw and a region D' whose boundary is the polygon P' obtained by P by replacing the sides $\overline{uv}$ and $\overline{vw}$ with the side $\overline{uw}$. So P' has $n-1 \geq 3$ sides and thus by induction can be triangulated. So the nontrivial case is when the interior of the line segment $\overline{uw}$ meets P. We claim that there is at least one vertex of P other than u, v, w either lying on $\overline{uw}$ or lying inside the triangle uvw. Indeed, starting from u and moving along $\overline{uw}$, consider the first intersection point (after u) of $\overline{uw}$ with P. If this point is a vertex of P, the claim has been established. Otherwise, this point is an interior point of a side s of P that is not parallel to $\overline{uw}$, so proceeding on s in one direction takes us inside the triangle uvw. This establishes the claim. Of all such vertices, choose t to maximize the distance from $\overline{uw}$. We claim that the line segment $\overline{tv}$ lies entirely inside D, i.e., is a diagonal. Assuming the claim for the moment, the argument finishes above, using the triangle tuv in place of uvw.

The argument for the claim is similar to what we've already done: if $\overline{tv}$ is not a diagonal, then travel along the line segment $\overline{tv}$ from t until we reach a side s' of P. If this intersection point is a vertex, then it is farther away from $\overline{uw}$ then t, giving a contradiction. If not, then if s' is parallel to $\overline{uv}$, then travelling in either direction along s' we reach a vertex that is farther away from $\overline{uv}$

than t, giving a contradiction. Finally, if s' is not parallel to $\overline{uv}$, then travelling along s' "away from $\overline{uv}'$" we reach a vertex that is farther away from $\overline{uv}$ than t. $\square$

Combining Theorem 4.11 and Lemma 4.12, we see that if U is simply connected and $f : U \to \mathbb{C}$ is holomorphic and $\gamma : [a, b] \to U$ is a simple closed polygonal path, then we can write $\int_\gamma f$ as a sum $\sum_{i=1}^n \int_{T_i} f$ where each T_i is a triangle in U which, since U is simply connected, means that the interior of T also lies in U. By Goursat's Lemma this gives $\int_\gamma f = 0$. By Theorems 4.6 and 4.8 it follows that for any closed path $\gamma : [a, b] \to U$ we have $\int_\gamma f = 0$.

4.2. **The Cauchy Integral Formula.**

Theorem 4.13. *Let $U \subset \mathbb{C}$ be a domain, and let $f : U \to \mathbb{C}$ be a holomorphic function. Let $a \in U$, and let $r > 0$ be such that the closed disk $\overline{B}_a(r)$ is contained in U. Let $C_r : [0, 2\pi] \to \mathbb{C}$ by $C_r(t) = a + re^{it}$. Then we have*

$$f(a) = \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(z)}{z-a} dz.$$

Proof. Step 1: In particular the conclusion implies that $\int_{C_r} \frac{f(z)}{z-a} dz$ is independent of r so long as r is small enough so that $\overline{B}_a(r)$ is contained in U. (This is already interesting!) First we will prove this independence of r and then we will exploit it to prove the result.

So let $0 < r_1 < r_2$ be such that $\overline{B}_a(r) \subset U$. We consider the following two contours: γ_1 begins with the straight line segment from $a+r_2$ (i.e., the rightmost point on the outer circle C_{r_2}) to $a+r_1$ (the rightmost point on the inner circle C_{r_1}), then takes the upper semicircular arc on the inner circle to $a-r_1$, then takes the straight line segment from $a-r_1$ to $a-r_2$, and then finally takes the upper semicircular arc on the outer circle back to $a+r_2$. Thus γ_1 is a simple closed curve that *does not* have a in its interior, so $\frac{f(z)}{z-a}$ is holomorphic on γ_1 and its interior, so by Cauchy-Goursat we have $\int_{\gamma_1} \frac{f(z)}{z-a} = 0$. The contour γ_2 begins at $a + r_2$ as well but first takes the bottom semicircular arc to $a - r_2$, then takes the straight line segment from $a - r_2$ to $a - r_1$, then takes the bottom semicircular arc to $a + r_1$, then finally takes the straight line segment from $a + r_1$ to $a + r_2$. For exactly the same reasons as above we have $\int_{\gamma_2} \frac{f(z)}{z-a} dz = 0$. So if γ is the path obtained by doing γ_1 followed by γ_2, certainly $\int_\gamma \frac{f(z)}{z-a} dz = 0$. But now examine what γ is overall: the two horizontal line segments are each traversed twice, once in each direction, so their contributions cancel out. Then the inner circle is traversed in the positive (counterclockwise) direction and the outer circle is traversed in the negative (clockwise) direction. So we get that

$$0 = \int_\gamma \frac{f(z)}{z-a} dz = \int_{C_{r_1}} \frac{f(z)}{z-a} dz - \int_{C_{r_2}} \frac{f(z)}{z-a} dz,$$

completing the proof of Step 1.

Step 2: We exploit the fact that no matter how small r is, we will get the same answer. Why does this help? Well, imagine that r is very, very tiny. Then in $\int_{C_r} \frac{f(z)}{z-a} dz$ the numerator $f(z)$ is a continuous function evaluated on the boundary of a very tiny disk, so therefore all of its values will be very close to the central point $f(a)$ of the disk. So the idea is as $r \to 0$ we should be able to replace the function $f(z)$ by the constant $f(a)$. It is a very easy calculation that

$$\int_{C_r} \frac{A}{z-a} dz = 2\pi i A$$

In fact, our very first line integral was the case $a = 0$, $r = 1$, $A = 1$ of this, and the general case is no harder. Try it!

To get a formal proof out of this, we write

$$\int_{C_r} \frac{f(z)}{z-a} dz = \int_{C_r} \frac{f(z)-f(a)}{z-a} dz + \int_{C_r} \frac{f(a)}{z-a} dz = \int_{C_r} \frac{f(z)-f(a)}{z-a} dz + 2\pi i f(a).$$

Thus it is enough to show that $\int_{C_r} \frac{f(z)-f(a)}{z-a} dz = 0$; again, by Step 1 we know that this quantity is independent of r, so it is enough to show that

$$\lim_{r \to 0} \int_{C_r} \frac{f(z)-f(a)}{z-a} dz = 0.$$

But this is not so bad: fix $\epsilon > 0$. Because f is differentiable at a, it is continuous at a, and thus for small enough r, z ranges over points that have distance r from a, so $|f(z) - f(a)| < \epsilon$. Also $|z - a| = r$, so the ML Inequality gives

$$\left| \int_{C_r} \frac{f(z)-f(a)}{z-a} dz \right| \leq \frac{\epsilon}{r} \cdot (2\pi r) = 2\pi\epsilon.$$

Thus the integral can be made arbitrarily small, and we're done! ☐

The following is an easy but striking consequence of Cauchy's Integral Formula.

Corollary 4.14. *(Mean Value Theorem) Let $U \subset \mathbb{C}$ be a domain, and let $f : U \to \mathbb{C}$ be a holomorphic function. Let $a \in U$, and let $R > 0$ be such that $\overline{B}_a(R) \subset U$. Then*

$$f(a) = \frac{1}{2\pi} \int_0^{2\pi} f(a + Re^{it}) dt.$$

Proof. If $\gamma : [0, 2\pi] \to \mathbb{C}$ by $\gamma(t) = a + Re^{it}$, then Cauchy's Integral Formula says

$$f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)dz}{z-a}.$$

We will get the result just by simplifying this expression. Indeed, we have $\gamma'(t) = iRe^{it}$, so

$$f(a) = \frac{1}{2\pi i} \int_0^{2i} \frac{f(a+Re^{it})}{(a+Re^{it}-a)} (iRe^{it} dt) = \frac{1}{2\pi} \int_0^{2\pi} f(a + Re^{it}) dt.$$ ☐

It is now time to begin the task of learning to appreciate Cauchy's Integral Formula. It is surprising, beautiful and useful, all wrapped into one. Let us first of all call attention to the fact that it says that the value of f at a point is determined by its values on any circle around that point. In particular, suppose f is entire – i.e., $U = \mathbb{C}$. Then $f(0)$ for instance is determined by the values of f on the circle of radius r for any r: in other words, knowing the values of f at points arbitrarily far away from 0 determine f at 0! So for instance, we immediately get the following:

Corollary 4.15. *Suppose $f : \mathbb{C} \to \mathbb{C}$ is an entire function. If there is $R > 0$ such that $f(z) = 0$ for all $|z| \geq R$, then f is identically 0.*

The point is that nothing like this is true for real functions. There are many many functions $f : \mathbb{R} \to \mathbb{R}$ that are infinitely differentiable, are strictly positive on $[-1, 1]$, but are identically 0 on $(-\infty, -2] \cup [2, \infty]$. Such functions are sometimes called "bump functions" and they are actually useful in certain parts of analysis and geometry.

The next thing to observe is that the proof did not make much use of the fact that the path of integration C_r was a circle. Suppose rather that we have a simple closed curve γ with image in U, such that a lies in the interior of γ, and such that γ is oriented in the positive direction – i.e., so that the interior of γ lies always to our left as we traverse γ. Then the proof works to show that $f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}dz = 0$ so long as, for all sufficiently small r, we can decompose the region between the inner circle C_r and the outer simple closed curve γ into a finite union of simple closed curves by adding in auxiliary piecewise smooth arcs. Though we do not want to prove it, it turns out that the conclusion is valid in general.

Theorem 4.16. *(Cauchy Integral Formula, v. 2) Let $U \subset \mathbb{C}$ be a domain, and let $f : U \to \mathbb{C}$ be a holomorphic function. Let $a \in U$, and let $\gamma : [a, b] \to U$ be a simple closed curve such that a lies in the interior of γ and the interior of γ is contained in U. Then*

$$f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)dz}{z-a}.$$

Here is another variant of the above reasoning: suppose we have simple closed curves γ_1 and γ_2, both positively oriented, with γ_1 lying inside the interior of γ_2. Again, so long as we can decompose the region lying between γ_1 and γ_2 into a finite union of simple closed curves by adding in auxiliary piecewise smooth arcs, then we get the following conclusion, which turns out to be valid in general.

Theorem 4.17. *(Cauchy Integral Theorem, v.2) Let γ_1 and γ_2 be positively oriented simple closed curves, iwth γ_1 lying inside the interior of γ_2. Let U be the domain of all z lying between γ_1 and γ_2 (i.e., exterior to γ_1 and interior to γ_2), and let $\overline{U}$ be U together with its boundary $\gamma_1 \cup \gamma_2$. If $f : \overline{U} \to \mathbb{C}$ is holomorphic, then we have*

$$\int_{\gamma_1} f = \int_{\gamma_2} f.$$

Example 4.18. *Let $f : \mathbb{C} \to \mathbb{C}$ be entire, and let γ be the* ***looped limacon*** *given by $r = \frac{1}{2} + \cos\theta$, oriented positively. Let $a = \frac{1}{4}$, which lies inside the inner loop. What is $\int_\gamma \frac{f(z)}{z-a}dz$? The point of course is that this is not a* simple *closed curve. However, it is obtained by concatenating two simple closed curves γ_1 and γ_2, each positively oriented and with a in the interior. So by Cauchy's Integral Formula we have*

$$\int_\gamma \frac{f(z)dz}{z-a} = \int_{\gamma_1} \frac{f(z)dz}{z-a} + \int_{\gamma_2} \frac{f(z)dz}{z-a} = 2\pi i f(a) + 2\pi i f(a) = 2(2\pi i f(a).$$

In the above example the outer "2" is there because the limacon "winds twice around a." More generally, if we draw a curve that spirals n times around a and then at the end comes back in to connect itself up, then similar arguments show that the integral will be $n(2\pi i f(a))$. If we change the orientation on the spiral curve, we get a negative integer n, and of course by going around a small disk disjoint from a we can get the integral to be 0. Is anything else possible?!? In fact not. We state the following result without proof.

Theorem 4.19. *Let U be a simply connected domain, let $f : U \to \mathbb{C}$ be a holomorphic function, and let $\gamma : [a, b] \to U$ be any piecewise smooth curve. Then $\frac{1}{2\pi i}\int_\gamma \frac{f(z)dz}{z-a}$ is an integer. It is called the* ***winding number*** *of γ about a.*

This result lies at the border of complex analysis and the important field of mathematics known as **algebraic topology**.

4.3. The Local Maximum Modulus Principle.

Exercise 4.1. *Let $g : [a, b] \to \mathbb{R}$ be continuous and non-negative, with maximum value M. Suppose $\int_a^b g = (b-a)M$. Show: $g(x) = M$ for all $x \in [a, b]$.*

Theorem 4.20. *(Local Maximum Modulus Principle) Let $a \in \mathbb{C}$, $R > 0$, and let $f : B_a(R) \to \mathbb{C}$ be holomorphic. Suppose that for all $z \in B_a(R)$ we have $|f(z)| \le |f(a)|$. Then f is constant.*

Proof. Step 1: Let $0 < r < R$. By Corollary 4.14 we have

$$f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a + re^{it})dt.$$

Since $|f|$ attains its maximum at a, we have

$$2\pi|f(a)| = |\int_0^{2\pi} f(a + re^{it})dt| \leq \int_0^{2\pi} |f(a + re^it)|dt \leq 2\pi|f(a)|.$$

Thus

$$\int_0^{2\pi} |f(a + re^{it})|dt = 2\pi|f(a)|.$$

The preceding exercise gives: $|f(z)| = |f(a)|$ for all $z = a + re^{it}$ and thus $|f|$ is constant on $B_a(R)$.
Step 2: We claim that for any domain $U \subset \mathbb{C}$ and holomorphic $f : U \to \mathbb{C}$, if $|f|$ is constant then f is constant. Write $f(z) = u(z) + iv(z)$, as usual. Thus our assumption is that there is $c \geq 0$ such that for all $z \in U$ we have

$$u^2(z) + v^2(z) = c.$$

Taking partial derivatives of the above equation we get

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0,$$

$$2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0.$$

Using the Cauchy-Riemann equations, the above equations imply

$$u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} = 0$$

and

$$u\frac{\partial u}{\partial y} + v\frac{\partial u}{\partial x} = 0.$$

In other words, we have the matrix equation

$$0 = \begin{bmatrix} \frac{\partial u}{\partial x} & -\frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial y} & \frac{\partial u}{\partial x} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

But this matrix has the familiar form $\begin{bmatrix} A & -B \\ B & A \end{bmatrix}$, so: if there is some $z \in U$ such that $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are not both zero, the matrix is invertible and thus $u(z) = v(z) = 0$, which implies that $u^2 + v^2 = c = 0$, so u and v are identically zero and f is constant. Otherwise $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are identically zero on U, so u is constant. The Cauchy-Riemann equations imply that $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ are also identically zero on U, so v is constant. □

Why is the above the "local" Maximum Modulus Principle? Well, suppose $U \subset \mathbb{C}$ is any domain, and let $f : U \to \mathbb{C}$ be holomorphic. Suppose that f assumes a maximum value at $a \in U$. Then in particular it assumes a maximum value in an open disk $B_a(R)$ for sufficiently small $R > 0$, and thus by Theorem f is constant on $B_a(R)$. Once we have proven the Identity Theorem – that says that it two holomorphic functions agree on a subset of U that has an accumulation point in U, they must be equal – we will be able to deduce that f is constant on U. Thus a nonconstant analytic function on a domain does not assume a maximum value.

Exercise 4.2. *(Local Minimum Modulus Principle) Let $a \in \mathbb{C}$, $R > 0$ and let $f : B_a(R) \to \mathbb{C}$ be holomorphic. Suppose that for all $z \in B_a(R)$ we have*

$$0 < |f(a)| \leq |f(z)|.$$

Show: f is constant.

4.4. **Cauchy's Integral Formula For Derivatives.** One remarkable consequence of Cauchy's Integral Formula is that it implies similar integral formulas for all the derivatives of a holomorphic function f: in particular it implies that any holomorphic function is infinitely differentiable!

Throughout this section, $U \subset \mathbb{C}$ is a domain, $f : U \to \mathbb{C}$ is a holomorphic function, z is a point of U, $r > 0$ is such that $\overline{B}_z(r)$ centered at r is contained in U, and $C_r(t) = z + re^{it}$.

We begin with the following lemma, an instance of "differentiation under the integral sign" that is possible to establish by relatively direct calculation.

Lemma 4.21. *For all $n \in \mathbb{Z}^+$, we have*

$$\frac{d}{dz}\int_\gamma \frac{f(w)dw}{(w-z)^n} = n\int_{C_r} \frac{f(w)dw}{(w-z)^{n+1}}.$$

Proof. We want to show that

$$\frac{d}{dz}\int_{C_r} \frac{f(w)dw}{(w-z)^n} - n\int_{C_r} \frac{f(w)dw}{(w-z)^{n+1}} = 0,$$

so let

$$L \coloneqq \frac{d}{dz}\int_{C_r} \frac{f(w)dw}{(w-z)^n} - n\int_{C_r} \frac{f(w)dw}{(w-z)^{n+1}}.$$

Then

$$L = \lim_{h\to 0} \frac{1}{h}\int_{C_r} \frac{f(w)}{(w-z-h)^n} - \frac{f(w)}{(w-z)^n}dw - n\int_{C_r} \frac{f(w)dw}{(w-z)^{n+1}}$$

$$= \lim_{h\to 0}\int_{C_r} f(w)\left(\frac{(w-z)^n - (w-z-h)^n}{(w-z-h)^n(w-z)^n h} - \frac{n}{(w-z)^{n+1}}\right)dw.$$

Put $s \coloneqq w - z$. Then

$$L = \lim_{h\to 0}\int_{C_r} f(w)\left(\frac{(s)^n - (s-h)^n}{(s-h)^n s^n h} - \frac{n}{s^{n+1}}\right)dw = \lim_{h\to 0}\int_{C_r} f(w)\left(\frac{s^{n+1} - s(s-h)^n - nh(s-h)^n}{h(s-h)^n s^{n+1}}\right)dw.$$

We claim that the numerator is of the form $h^2 Q(s,h)$, where $Q(s,h)$ is a polynomial in s and h. To see this, we expand out the binomials and write $O(h^2)$ for expressions that are sums each term of which is divisible by h^2. We get:

$$s^{n+1} - s(s-h)^n - nh(s-h)^n = s^{n+1} - s\left(s^n - \binom{n}{1}s^{n-1}h + O(h^2)\right) - nh\left(s^n - \binom{n}{1}s^{n-1}h + O(h^2)\right)$$

$$= s^{n+1} - s^{n+1} + nhs^n + O(h^2) - nhs^n - +n^2 s^{n-1}h^2 + O(h^2) = n^2 s^{n-1}h^2 + O(h^2) + O(h^2) = O(h^2).$$

Thus $L = 0$ iff

$$\lim_{h\to 0} h\left|\int_{C_r} \frac{f(w)Q(s,h)dw}{(s-h)^n s^{n+1}}\right| = 0.$$

Thus it is enough to show that $\left|\int_{C_r} \frac{f(w)Q(s,h)dw}{(s-h)^n s^{n+1}}\right|$ stays bounded as $h \to 0$, since then its product with h approaches 0. Fix one $r_0 > 0$ such that $\overline{B}_z(r_0) \subset U$. Then the function $fQ(s,h)$ is continuous on $\mathbb{C}$ hence is bounded on C_r. Moreover we have $|s| = |w - z| = r$, and for sufficiently small h we have $|s - h| \geq \frac{r}{2}$. Therefore the ML-Inequality gives

$$\left| \int_{C_r} \frac{f(w)Q(s,h)dw}{(s-h)^n s^{n+1}} \right| \leq 2\pi r \frac{M}{(r/2)^n r^{n+1}},$$

which is indeed bounded independently of h as $h \to 0$. So $L = 0$, completing the proof. □

Theorem 4.22. *(Cauchy Integral Formula for Derivatives) For all $n \in \mathbb{N}$, $f^{(n)}(z)$ exists and is given by the following equation:*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{C_r} \frac{f(w)dw}{(w-z)^{n+1}}. \tag{12}$$

Proof. We go by induction on n. The base case $n = 0$ is Cauchy's Integral Formula. Now let $n \in \mathbb{N}$, and suppose that (12) holds for n. Then using Lemma 4.21 we get

$$f^{(n+1)}(z) = \frac{d}{dz} f^{(n)}|_z = \frac{n!}{2\pi i} \frac{d}{dz} \int_{C_r} \frac{f(w)dw}{(w-z)^{n+1}} = \frac{(n+1)!}{2\pi i} \int_{C_r} \frac{f(w)dw}{(w-z)^{n+2}},$$

completing the induction step. □

We have the following remarkable consequence.

Corollary 4.23. *If f is holomorphic on U then it is infinitely differentiable, and every derivative is again holomorphic on U.*

Exercise 4.3. *Let $f(z) = u(z) + iv(z)$ be a holomorphic function on U. Show that $u : U \to \mathbb{R}$ and $v : U \to \mathbb{R}$ are smooth functions: that is, partial derivatives of all orders exist.*

Exercise 4.4. *Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \begin{cases} x^{2018} & x \geq 0 \\ -x^{2018} & x < 0 \end{cases}$. Show that $f^{(2017)}$ exists but $f^{(2018)}$ does not.*

4.5. **Morera's Theorem.**

Theorem 4.24. *(Morera[10]'s Theorem) Let $U \subset \mathbb{C}$ be a domain, and let $f : U \to \mathbb{C}$ be a function. The following are equivalent:*
(i) f is holomorphic on U.
(ii) For every simple closed curve γ such that γ and its interior lie in U, we have $\int_\gamma f = 0$.
(iii) For every triangle T that is contained, along with its interior, in U, we have $\int_T f = 0$.

Proof. (i) $\implies$ (ii) by the Cauchy-Goursat Theorem.
(ii) $\implies$ (iii) is immediate.
(iii) $\implies$ (i): By Lemma 4.12, every simple polygonal closed curve can be triangulated, and thus the hypothesis implies that $\int_\gamma f = 0$ for every simple polygonal closed path. By Theorem Y.Y this implies that there is $F : U \to \mathbb{C}$ such that $F' = f$. Clearly F is holomorphic, and it follows from Theorem 4.22 that the derivative of every holomorphic function is holomorphic, hence also f is holomorphic. □

One should think of Morera's Theorem as saying that holomorphic functions are precisely the analogue of irrotational vector fields. Another way to say it is that they are precisely the "locally conservative" functions. Being holomorphic is a *local property* of a function – i.e., it depends only on the values of a function in an arbitrarily small open disk centered at the point. Similarly, being an irrotational vector field is a local property, and Green's Theorem implies that an irrotational vector field is conservative in any open disk around each point, hence locally conservative.

[10]Giacinto Morera, 1856-1909, was an Italian engineer and mathematician.

4.6. **Cauchy's Estimate.**

Theorem 4.25. *(Cauchy's Estimate) Let $f : \overline{B}_a(R) \to \mathbb{C}$ be holomorphic, and let M be the supremum of $|f(z)|$ on the circle C_R of radius R centered at a. Then for all $n \in \mathbb{N}$, we have*

$$|f^{(n)}(a)| \leq \frac{n!M}{R^n}.$$

Proof. Let C_R be the boundary circle of $\overline{B}_a(R)$. We use Cauchy's Integral Formula for Derivatives and the ML-Inequality:

$$|f^{(n)}(a)| = |\frac{n!}{2\pi i}\int_{C_R} \frac{f(w)dw}{(w-a)^{n+1}}| \leq \frac{n!}{2\pi}\frac{M}{R^{n+1}}(2\pi R) = \frac{n!M}{R^n}. \qquad \square$$

4.7. **Liouville's Theorem.**

Theorem 4.26. *(Liouville[11]'s Theorem) Let $f : \mathbb{C} \to \mathbb{C}$ be entire and bounded. Then f is constant.*

Proof. Suppose $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Then Cauchy's Estimate with $n = 1$ gives

$$|f'(a)| \leq \frac{M}{R}.$$

But we are allowed to take R as large as we want, so this implies $|f'(a)| = 0$. Since a was arbitrary, this means that f' is identically 0. By Theorem 2.14, this implies that f is constant. $\square$

Again this is a striking difference from the real variable case, where for instance the infinitely differentiable functions $\sin x$ and $\cos x$ are bounded by 1 on $\mathbb{R}$.

Exercise 4.5. *a) Let $R > 0$. Find an explicit complex number z_R (in terms of R, of course) such that $|\cos z_R| \geq R$.*
b) Let $R > 0$. Find an explicit complex number z_R (in terms of R, of course) such that $|\sin z_R| \geq R$.
c) Let $R > 0$. Is there a complex number z such that $|\cos z| \geq R$ and $|\sin z| \geq R$?

The technique of proof of Liouville's Theorem also works to establish several generalizations.

Theorem 4.27. *Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function.*
a) (Generalized Liouville) Suppose there are $\alpha \in [0, 1)$ and $C > 0$ such that $|f(z)| \leq C|z|^\alpha$ for all $z \in \mathbb{C}$. Then f is constant.
b) (Polynomial Liouville) Let $g(z)$ be a polynomial of degree $d \geq 0$. Suppose that $|f(z)| \leq |g(z)|$ for all $z \in \mathbb{C}$. Then $f(z)$ is itself a polynomial of degree at most d.

Proof. a) Let $a \in \mathbb{C}$, let $R > 0$ and let $M(R, a)$ be the maximum value of f on the circle $C(R, a)$ of radius R centered at a. Cauchy's Estimate with $n = 1$ gives

$$|f'(a)| \leq \frac{M(R, a)}{R}.$$

If $z \in C(R, a)$, then $|z| \leq R + |a|$, so $M(R, a) \leq C(R + |a|)^\alpha$, so for sufficiently large R we have

$$|f'(a)| \leq \frac{C(R + |a|)^\alpha}{R} \leq \frac{C(2R)^\alpha}{R}.$$

Letting $R \to \infty$ shows $f'(a) = 0$, so f is constant by the Zero Velocity Theorem.
b) Let $a \in \mathbb{C}$, let $R > 0$, and again let $M(R, a)$ be the supremum of f on the circle $C(R, a)$ of radius R centered at a. Cauchy's Estimate with $n = d + 1$ gives

$$|f^{(d+1)}(a)| \leq \frac{(d + 1)!M(R, a)}{R^{d+1}}.$$

[11] Joseph Liouville, 1809-1882, was a versatile French mathematician, working in analysis, geometry, mathematical physics and number theory, among other areas. Why this theorem is named after him since it follows in a few lines from a result of Cauchy is mysterious to me.

If $g(z) = a_d z^d + \ldots + \ldots + a_1 z + a_0$, then for sufficiently large $|z|$ we have

$$|g(z)| \leq |2a_d||z|^d.$$

If $z \in C(R, a)$, then again $|z| \leq R + |a|$, so

$$|f(z)| \leq |g(z)| \leq |2a_d|(R + |a|)^d.$$

Thus for sufficiently large R we have

$$M(R, a) \leq |2a_d|(R + |a|^d) \leq 2^{d+1}|a_d|R^d,$$

so

$$|f^{(d+1)}(a)| \leq \frac{(d+1)!2^{d+1}|a_d|R^d}{R^{d+1}} = \frac{(d+1)!2^{d+1}|a_d|}{R}.$$

The right hand side approaches 0 as $R \to \infty$, so $f^{(d+1)}$ is identically 0. By Exercise 2.7 it follows that f is a polynomial of degree at most d. □

4.8. **The Fundamental Theorem of Algebra.**

Lemma 4.28. *Let $f : \mathbb{C} \to \mathbb{C}$ be a continuous function such that $\lim_{z \to \infty} f(z) = L \in \mathbb{C}$. Then f is bounded.*

Proof. Let $R \geq 0$ be such that for $|z| \geq R$ we have $|f(z) - L| < 1$; for such z we have $|f(z)| \leq |L| + 1$. Since f is continuous and $\overline{B}_0(R)$ is closed and bounded, by Theorem 1.28 there is $M > 0$ such that $|f(z)| \leq M$ for all z with $|z| \leq R$. It follows that f is bounded on $\mathbb{C}$ by $\max(L + 1, M)$. □

Theorem 4.29. *(Fundamental Theorem of Algebra) Let $f(z)$ be a polynomial of degree $n \geq 1$ with complex coefficients. There are $r_1, \ldots, r_n \in \mathbb{C}$ and $\alpha \in \mathbb{C} \setminus \{0\}$ such that*

$$f(z) = \alpha(z - r_1) \cdots (z - r_n).$$

Proof. Step 1: We show that there is $z \in \mathbb{C}$ such that $f(z) = 0$. Seeking a contradiction, we assume not: then $g(z) := \frac{1}{f(z)}$ is an entire function. Moreover, since f has positive degree, we have $\lim_{z \to \infty} g(z) = 0$. By Lemma 4.28 we have that g is bounded, and then g is constant by Liouville's Theorem, contradicting the fact that g is nonzero and $\lim_{z \to \infty} g(z) = 0$.

Step 2: Thus there is $r_1 \in \mathbb{C}$ such that $f(r_1) = 0$. By the Root-Factor Theorem from high school algebra, we may write $f(z) = (z - r_1)f_2(z)$ for a polynomial $f_2(z)$ of degree $n - 1$. Applying Step 1 to f_2 we continue in this manner, eventually getting $f(z) = (z - r_1) \cdots (z - r_{-}n)f_n(z)$ with $f_n(z)$ of degree 0. That is, $f_n(z)$ is a nonzero constant α and $f(z) = \alpha(z - r_1) \cdots (z - r_n)$. □

4.9. **Logarithms.**

Theorem 4.30. *Let $U \subset \mathbb{C}$ be a simply connected domain. Let $g : U \to \mathbb{C} \setminus \{0\}$ be a holomorphic function. Then there is a holomorphic function $f : U \to \mathbb{C}$ such that $g = e^f$.*

Proof. Fix $a \in U$. Since $e^z = e^x e^{iy}$ and $e^x : \mathbb{R} \to (0, \infty)$ is surjective, there is $c \in \mathbb{C}$ such that $e^c = f(a)$. We define

$$g(z) := c + \int_a^z \frac{f'(w)dw}{f(w)}.$$

Here the notation means that we integrate along any path from a to z: because f'/f is holomorphic and U is simply connected, by Cauchy-Goursat the integral is independent of the choice of path, and by Theorem 4.8 we have $g'(z) = \frac{f'(z)}{f(z)}$ for all $z \in U$.

We claim that g is the desired function. We show that $g = e^f$ in a slightly sneaky way: let

$$G : U \to \mathbb{C}, z \mapsto f(z)e^{-g(z)}.$$

Then

$$G'(z) = \frac{f'(z)}{e^{g(z)}} + \frac{f(z)}{e^{g(z)}}(-g'(z)) = \frac{f'(z)}{e^{g(z)}} - \frac{f(z)}{e^{g(z)}}\frac{f'(z)}{f(z)} = 0.$$

Since U is connected, by Theorem 2.14 the function G is constant. Evaluating at $z = a$ we get that

$$\forall z \in U,\ \frac{f(z)}{e^{g(z)}} = \frac{f(a)}{e^{g(a)}} = \frac{f(a)}{e^c} = 1.$$

That is, for all $z \in U$ we have $f(z) = e^{g(z)}$. $\square$

If $U \subset \mathbb{C}$ is a domain, then a **branch of the logarithm** is a continuous function $L : U \to \mathbb{C}$ such that $e^{L(z)} = z$ for all $z \in \mathbb{C}$. Above we saw that there is no branch of the logarithm defined on $\mathbb{C} \setminus \{0\}$. On the other hand, if U is a simply connected domain with $0 \notin U$, then applying Theorem 4.30 with $g(z) = z$, we get that there is a branch of the logarithm defined on U.

Now we consider the following function: for $z \in \mathbb{C} \setminus \{0\}$, let $\operatorname{Arg}(z)$ to be the argument of z taken to lie in the interval $(-\pi, \pi]$, sometimes called the **principal argument**. We define

$$\operatorname{Log}(z) := \log|z| + i \operatorname{Arg}(z).$$

The reason for this definition is that for all $z \in \mathbb{C} \setminus \{0\}$ we have

$$e^{\operatorname{Log}(z)} = e^{\log|z|}e^{i\operatorname{Arg}(z)} = |z|e^{i\operatorname{Arg}(z)} = z.$$

However:

Exercise 4.6. *Show that* Log *is continuous at z iff z is not a real number $x \leq 0$. (Suggestion: the function* $\log|z|$ *is continuous on $\mathbb{C} \setminus \{0\}$, so it comes down to* $\operatorname{Arg}(z)$*.)*

It follows from this exercise that Log is a branch of the logarithm on the smaller domain $U = \mathbb{C} \setminus \{x \in \mathbb{R} \mid x \leq 0\}$: notice that this smaller domain is simply connected.

Exercise 4.7. *a) Show: for $z = x + iy$, we have*

$$\operatorname{Log}(e^z) = x + i\operatorname{Arg}(e^{iy}).$$

b) Deduce that $\operatorname{Log}(e^z) = z$ *iff* $\Im(z) \in (-\pi, \pi]$.
c) Let $V = \{z \in \mathbb{C} \mid \Im(z) \in (-\pi, \pi)\}$ and let $U = \mathbb{C} \setminus \{x \in \mathbb{R} \mid x \leq 0\}$. Show that $\exp : V \to U$ *and* $\operatorname{Log} : U \to V$ *are mutually inverse bijections.*

Using these exercises, we deduce:

Theorem 4.31. *The function* $\operatorname{Log} \mathbb{C} \setminus \{x \in \mathbb{R} \mid x \leq 0\} \to \mathbb{C}$ *is holomorphic.*

Proof. We apply the Inverse Function Theorem (Theorem 2.10) with $U = \{z \in \mathbb{C} \mid \Im(z) \in (-\pi, \pi)\}$, $V = \mathbb{C} \setminus (-\infty, 0]$, $f(z) = e^z$ and $g(z) = \operatorname{Log} z$. By the preceding exercise, f is bijective and g is its inverse function, so the Inverse Function Theorem tells us that g is holomorphic. $\square$

Exercise 4.8. *Show that if $U \subset \mathbb{C}$ is a domain and $L : U \to \mathbb{C}$ is a branch of the logarithm, then L is holomorphic on U.*
(Suggestion: Adapt the above argument. You will need to shrink U so that the exponential function is injective on $L(U)$.)

For $a \in \mathbb{C} \setminus \{0\}$ and $b \in \mathbb{C}$, we define the **principal value**

$$a^b := e^{b\operatorname{Log} a}.$$

For now we just warn that these principal values must be interpreted carefully; we *do not claim* that it satisfies all the usual laws of exponents.

4.10. **Harmonic functions II.** The next result asserts that every harmonic function $u : U \to \mathbb{R}$ on a simply connected domain has a harmonic conjugate, i.e., a harmonic function $v : U \to \mathbb{R}$ such that $f = u + iv$ is holomorphic on U. Recall in fact that the Cauchy-Riemann equations show that the real and imaginary parts of any holomorphic function are harmonic, so it is enough to show that $u = \Re f$ for some holomorphic f.

Theorem 4.32. *Let $U \subset \mathbb{C}$ be a simply connected domain, and let $u : U \to \mathbb{R}$ be harmonic. Then there is a holomorphic function $f : U \to \mathbb{C}$ such that $u = \Re(f)$.*

Proof. Step 1: $g : U \to \mathbb{C}$ by $g(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$. We will show that g satisfies the Cauchy-Riemann equations. Since u is harmonic, g has continuous first partial derivatives, so by Theorem 2.17, g is holomorphic on U. Indeed, we have $\Re(g) = \frac{\partial u}{\partial x}$ and $\Im(g) = -\frac{\partial u}{\partial y}$, so

$$\frac{\partial \Re(g)}{\partial x} = \frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} = \frac{\partial \Im(g)}{\partial y},$$

the second equality holding because u is harmonic; and

$$\frac{\partial \Re(g)}{\partial y} = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} = -\frac{\partial \Im(g)}{\partial x}.$$

Step 2: Because $g : U \to \mathbb{C}$ is holomorphic and U is simply connected, there is a function $G : U \to \mathbb{C}$ with $G' = g$. Let us write $G(z) = a(z) + ib(z)$. Then

$$\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = g(z) = h'(z) = \frac{\partial h}{\partial x} = \frac{\partial a}{\partial x} + i\frac{\partial b}{\partial x} = \frac{\partial a}{\partial x} - i\frac{\partial a}{\partial y},$$

so we have

$$\frac{\partial a}{\partial x} = \frac{\partial u}{\partial x} \tag{13}$$

and

$$\frac{\partial u}{\partial y} = \frac{\partial a}{\partial y}. \tag{14}$$

Equation (13) implies that $\frac{\partial}{\partial x}(u - a) = 0$, and thus there is a function $c(y)$ such that

$$c(y) = u(x, y) - a(x, y).$$

Differentiating both sides of this equation with respect to y and using (14) we get

$$c'(y) = \frac{\partial u}{\partial y} - \frac{\partial a}{\partial y} = 0.$$

Thus in fact $c(y) = c$ is a constant, so $G(z) + c$ is holomorphic and

$$u(z) = a(z) + c = \Re(G(z) + c),$$

completing the proof. $\square$

If we can explicitly write down the antiderivative G of g, then we can explicitly find the harmonic conjugate v. The following is a simple example.

Example 4.33. *The function $u : \mathbb{R}^2 \to \mathbb{R}$ given by $u(x, y) = xy$ is harmonic. Let*

$$g(z) := \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = y - ix.$$

We recognize $g(z) = -i(x + iy) = -iz$, so an antiderivative is

$$G(z) = \frac{-i}{2}z^2 = \frac{-i}{2}(x + iy)^2 = \frac{-i}{2}(x^2 - y^2 + 2xyi) = xy + i\frac{y^2 - x^2}{2},$$

so the harmonic conjugate of u is $v(x,y) = \frac{y^2-x^2}{2}$.

Corollary 4.34. *Let $U \subset \mathbb{R}^2$ be a domain. Then every harmonic function $u : U \to \mathbb{R}$ is infinitely differentiable.*

Proof. The partial derivatives of u at a point depend only on the values of u in a small disk around the point, so we reduce to the case in which U is an open disk, which is simply connected. Then u is the real part of a holomorphic function, so the mixed partial derivatives of all orders exist. $\square$

From the perspective of real analysis, Corollary 4.34 is a very striking result. By definition harmonic functions are C^2 – i.e., all second partial derivatives exist and are continuous. Of course a real function that is C^2 need not be C^3, let alone infinitely differentiable. However satisfying the differential equation $\Delta u = 0$ forces the additional smoothness. Results of this kind that assert that a weak smoothness property together with solving a differential equation satisfies a stronger smoothness property are called **regularity theorems**. The result about harmonic functions is in fact a case of a more general theorem: there is a class of PDEs called **elliptic** and an **Elliptic Regularity Theorem** concerning solutions of elliptic PDEs.

Theorem 4.35. *(Mean Value Theorem for Harmonic Functions) Let $U \subset \mathbb{R}^2$ be a domain and $h : U \to \mathbb{R}$ be a harmonic function. Let $a \in U$ and let $r > 0$ be such that $\overline{B}_a(r) \subset U$. Then:*

$$h(a) = \frac{1}{2\pi}\int_0^{2\pi} u(a + re^{it})dt. \tag{15}$$

Proof. The idea of the proof is to combine Theorems 4.14 and 4.32. There is one minor technicality: we to apply Theorem 4.32 we need a simply connected *domain*, i.e., an open set. In fact there is $R > r$ such that $\overline{B}_a(r) \subset B_a(R)$. On first pass the reader may just want to add the existence of such an R as a hypothesis to the theorem. Anyway, here is a proof: if $U = \mathbb{R}^2$ then this is clear. Otherwise, for every $z \in \overline{B}_a(r)$ we let $d(z)$ be the distance from z to the boundary of U. In any metric space, the distance from a varying point to a fixed closed set A is well-defined and continuous, and when restricted to points lying outside A is strictly positive. Applying this with A as the boundary of U we get a continuous function $d : \overline{B}_a(r) \to (0,\infty)$. Since $\overline{B}_a(r)$ is closed and bounded, this function attains its minimum value, i.e., there is $\delta > 0$ such that $d(z) \geq \delta$ for all $z \in \overline{B}_a(r)$. Taking $R := r + \delta$, it follows that $B_a(R) \subset U$.

Having established this, we apply Theorem 4.32 to h on $B_a(r+\delta)$: there is a holomorphic $f : B_a(R) \to U$ with $\Re(f) = h$. By Theorem 4.14 we have

$$f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a + re^{it})dt.$$

Taking the real part of both sides gives (15). $\square$

Corollary 4.36. *A harmonic function $h : U \to \mathbb{R}$ cannot have a strict local extremum at any point of U.*

Proof. It is enough to show that h cannot have a strict local maximum at any $a \in U$; if so, applying this to $-h$ in place of h shows that h cannot have a strict local minimum. To say that h has a strict local maximum at a means that there is some $r > 0$ such that $\overline{B}_r(a) \subset U$ and for all $z \in \overline{B}_R(a) \setminus \{a\}$ we have $h(a) > h(z)$. Because h is continuous and the boundary circle $C_r(a)$ is closed and bounded, h assumes a maximum value M on $C_r(a)$ and thus $M < h(a)$. But then

$$h(a) = \frac{1}{2\pi}\int_0^{2\pi} u(a + re^{it})dt \leq \frac{1}{2\pi}\int_0^{2\pi} M dt = \frac{2\pi M}{2\pi} = M < h(a),$$

a contradiction. $\square$

Corollary 4.37. *If $f : U \to \mathbb{C} \setminus \{0\}$ is holomorphic, then $h = \log|f|$ is harmonic.*

Proof. Because being harmonic is a local property of f, we may assume that U is an open disk and thus simply connected. We may thus write $f = e^g$, and then for all $z \in U$ we have

$$\log|f(z)| = \log|e^{g(z)}| = \log e^{\Re(g(z))} = \Re(g(z)).$$

Being the real part of a holomorphic function, $\log|f(z)|$ is harmonic. $\square$

5. Some integrals

Example 5.1. *Let $a > 1$. We evaluate $I = \int_0^{2\pi} \frac{dt}{a+\cos t}$. We have*

$$I = \int_0^{2\pi} \frac{dx}{a + \frac{e^{ix}+e^{-ix}}{2}} = 2\int_0^{2\pi} \frac{e^{ix}dx}{e^{2ix} + 2ae^{ix} + 1}.$$

Put $z = e^{ix}$, so $dz = ie^{ix}dx$. Thus

$$I = \frac{2}{i}\int_{C_0(1)} \frac{dz}{z^2 + 2az + 1}.$$

The roots of $z^2 + 2az + 1$ are $r_1 = -a - \sqrt{a^2-1}$ and $r_2 = -a + \sqrt{a^2-1}$. Since $r_1 < -a < -1$, we have $|r_1| > 1$. On the other hand, we have

$$r_1 r_2 = (-a)^2 - (\sqrt{a^2-1})^2 = 1,$$

so $|r_2| < 1$. So by Cauchy's Integral Formula, we have

$$I = \frac{2}{i}\int_{C_0(1)} \frac{\frac{dz}{z-r_1}}{z - r_2} = \frac{2}{i}2\pi i \frac{1}{r_2 - r_1} = \frac{4\pi}{2\sqrt{a^2-1}} = \frac{2\pi}{\sqrt{a^2-1}}.$$

This solution is taken from `https://math.stackexchange.com/questions/134577/how-do-you-integrate-int` *In fact that question asks for the* antiderivative *instead, and several answers there explain how this can be done using the (in)famous* Weierstrass substitution.

Example 5.2. *We evaluate*

$$I = \int_{-\infty}^{\infty} \frac{dx}{x^2+1}.$$

Let $R > 0$. Let γ_R be the contour which begins with a straight line segment from $-R$ to R and then follows with the semicircular arc in the upper half plane from R to $-R$. First we claim that

$$\lim_{R\to\infty} \int_{\gamma_R} \frac{dz}{z^2+1} = \int_{-\infty}^{\infty} \frac{dx}{x^2+1}.$$

The integral along the line segment from $-R$ to R is just $\int_R^R \frac{dx}{x^2+1}$, so the limit of this as R approaches infinity is $\int_{-\infty}^{\infty} \frac{dx}{x^2+1}$. (This improper integral is absolutely convergent so coincides with its principal value.) Let $C_R : [0,\pi] \to \mathbb{C}$, $t \mapsto Re^{it}$ parameterize the semicircular arc. So to establish the claim we need to show that

$$\lim_{R\to\infty} \int_{C_R} \frac{dz}{z^2+1} = 0.$$

For this the ML-Inequality suffices: for sufficiently large R, $Q(z) = z^2 + 1$ is at least $\frac{1}{2}R^2$ in absolute value, so $|\frac{1}{z^2+1}| \leq \frac{2}{R^2}$. Since the length of the semicircular arc is πR, we get

$$\left|\int_{C_R} \frac{dz}{z^2+1}\right| \leq \frac{2\pi R}{2R^2} = \frac{\pi}{R} \to 0.$$

Since $\frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)}$, *we put* $g(z) = \frac{1}{z+i}$. *Then* $g(z)$ *is holomorphic except at* i, *hence on* γ_R *and its interior. Applying Cauchy's Integral formula, we get*

$$I = 2\pi i g(i) = 2\pi i\left(\frac{1}{2i}\right) = \pi.$$

Comments: a) that we can also do this via real variable methods. Indeed

$$I = \lim_{R\to\infty}\int_{-R}^{R}\frac{dx}{x^2+1} = \lim_{R\to\infty}\arctan x|_{R}^{R} = \arctan(\infty) - \arctan(-\infty) = \frac{\pi}{2} - \frac{-\pi}{2} = \pi.$$

b) The first part of the argument works when integrating anything of the form $\frac{f(z)}{Q(z)}$ *where* f *is holomorphic and bounded for all* z *with* $\Re(z) \geq 0$ *and the denominator is a polynomial* $Q(z)$ *of degree at least* 2.

We pause to develop the partial fractions decomposition of a proper rational function over $\mathbb{C}$. (This is similar to but *easier than* the corresponding theory over $\mathbb{R}$ owing to the absence of irreducible quadratic polynomials over $\mathbb{C}$.) Let $\frac{P(z)}{Q(z)}$ be a proper rational function: that is, $P(z)$ and $Q(z)$ are polynomials and the degree of P is less than the degree of Q. By the Fundamental Theorem of Algebra we may write

$$Q(z) = (z-r_1)^{a_1}\cdots(z-r_k)^{a_k}$$

for distinct complex numbers $r_1,\ldots,r_k$ and positive integers $a_1,\ldots,a_k$. Then the degree of Q is $a_1+\ldots+a_k = n$, say. We claim that there are unique complex numbers $C_1,\ldots,C_n$ such that

$$\frac{P(z)}{Q(z)} = \frac{C_1}{z-r_1} + \frac{C_2}{(z-r_1)^2} + \ldots + \frac{C_{a_1}}{(z-r_1)^{a_1}} + \ldots + \frac{C_{a_1+\ldots+a_{k-1}+1}}{z-r_k} + \ldots + \frac{C_n}{(z-r_k)^{a_k}}.$$

We will prove this by linear algebra. First, we observe that the set of proper rational functions that can be written in the form $\frac{P(z)}{Q(z)}$ is a vector space over $\mathbb{C}$. Since one basis is given by $\{\frac{z^i}{Q(z)}\}_{0\leq i\leq n-1}$, the space has dimension n. The partial fractions decomposition is precisely the assertion that $\frac{1}{z-r_1},\ldots,\frac{1}{(z-r_1)^{a_1}},\ldots,\frac{1}{(z-r_k)^{a_k}}$ is also a basis for the same vector space (by putting all these terms over a common denominator we see that they are indeed proper rational functions of the form $\frac{P(z)}{Q(z)}$). Because the number of elements is again n, it is enough to show that these elements are linearly independent over $\mathbb{C}$, i.e., if

$$\frac{C_1}{z-r_1} + \frac{C_2}{(z-r_1)^2} + \ldots + \frac{C_{a_1}}{(z-r_1)^{a_1}} + \ldots + \frac{C_{a_1+\ldots+a_{k-1}+1}}{z-r_k} + \ldots + \frac{C_n}{(z-r_k)^{a_k}} = 0, \tag{16}$$

then $C_i = 0$ for all $1 \leq i \leq n$. To see this, multiply (16) through by $(z-r_1)^{a_1}$. Having done that, every term but the term with coefficient C_{a_1} has $(z-r_1)$ as a factor, and that latter term is just C_{a_1}, so evaluating at $z = r_1$ we get $C_{a_1} = 0$. Having learned this we can now repeat the same argument by multiplying through by $(z-r_1)^{a_1-1}$ to get $C_{a_1-1} = 0$, and continuing in this way we get $C_1 = \ldots = C_{a_1} = 0$. Repeating this argument for the roots r_2 through r_k eventually gives that all coefficients are 0.

Example 5.3. *We evaluate*

$$I = \int_{-\infty}^{\infty}\frac{dx}{x^4+1}.$$

By Remark b) of the above example, if γ_R *is the upper semicircular contour from* $-R$ *to* R *followed by the straight line segment from* $-R$ *to* R, *we have*

$$I = \lim_{R\to\infty}\int_{\gamma_R}\frac{dz}{z^4+1}.$$

The roots of z^4+1 *are the fourth roots of* -1*, namely*

$$r_1 = e^{i\pi/4},\ r_2 = e^{3\pi i/4},\ r_3 = e^{5\pi i/4},\ r_4 = e^{7\pi i/4}.$$

We can do a partial fractions decomposition

$$\frac{1}{z^4+1} = \frac{A}{z-r_1} + \frac{B}{z-r_2} + \frac{C}{z-r_3} + \frac{D}{z-r_4}.$$

Because r_3 *and* r_4 *lie in the lower half plane so outside of* γ_R*, we have* $\int_{\gamma_R} \frac{C}{z-r_3} + \int_{\gamma_R} \frac{D}{z-r_3} = 0$*, and thus*

$$I = \lim_{R\to\infty} \int_{\gamma_R} \frac{A}{z-r_1} + \int_{\gamma_R} \frac{B}{z-r_2} = 2\pi i(A+B).$$

Now we find A *and* B*: clearing denominators and evaluating at* $z = r_1$ *gives*

$$1 = A(r_1-r_2)(r_1-r_3)(r_1-r_4),\ \text{so}\ A = \frac{1}{(r_1-r_2)(r_1-r_3)(r_1-r_4)},$$

while clearing denominators and evaluating at $z = r_2$ *gives similarly*

$$B = \frac{1}{(r_2-r_1)(r_2-r_3)(r_2-r_4)}.$$

Example 5.4. *We evaluate* $I = \int_{-\infty}^{\infty} \frac{\cos x dx}{x^2+1}$*. First we observe that* I *is the real part of* $J = \int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+1}$*. Now here is a key observation: for all* $z = x+iy \in \mathbb{C}$*, if* $y \geq 0$*, then*

$$|e^{iz}| = |e^{i(x+iy)}| = e^{-y}|e^{ix}| = e^{-y} \leq 1.$$

In other words, the function e^{iz} *is holomorphic and bounded on the upper half plane. As noted above, this implies that*

$$J = \lim_{R\to\infty} \int_{C_R} \frac{e^{iz}}{z^2+1},$$

where C_R *is the semicircular contour as above. Writing* $\frac{e^{iz}}{z^2+1} = \frac{e^{iz}/(z+i)}{z-i}$ *and applying the Cauchy Integral Formula with* $g(z) = e^{iz}/(z+i)$*, we get*

$$J = 2\pi i g(i) = (2\pi i)e^{i\cdot i}/(i+i) = \frac{\pi}{e}.$$

Thus

$$I = \int_{-\infty}^{\infty} \frac{\cos x dx}{x^2+1} = \Re(\frac{\pi}{e}) = \frac{\pi}{e}.$$

a) It also follows that $\int_{-\infty}^{\infty} \frac{\sin x}{x^2+1} = 0$*. However, we should have known this already: weT are integrating an odd function from* $-\infty$ *to* ∞*, so of course we get zero!*

Using the methods we have developed, if γ is any simple closed curve, f is a function that is defined and holomorphic on γ and its interior and $Q(z)$ is any polynomial, then we can compute $\int_\gamma \frac{f(z)dz}{Q(z)}$. We give one (relatively simple) example.

Example 5.5. *Let* $\gamma : [0, 2\pi] \to \mathbb{C}$ *by* $\gamma(t) = 3e^{it}$ *be the circle of radius three centered at* 0*, positively oriented. Let* $f : \mathbb{C} \to \mathbb{C}$ *be any entire function. We will compute*

$$I = \int_\gamma \frac{f(z)dz}{(z-1)^2(z-2)}.$$

First we do a partial fractions decomposition:

$$\frac{1}{(z-1)^2(z-2)} = \frac{A}{z-1} + \frac{B}{(z-1)^2} + \frac{C}{z-2}.$$

Clearing denominators we get

$$1 = (z-1)(z-2)A + (z-2)B + (z-1)^2 C. \tag{17}$$

Evaluating at $z = 1$ gives $B = -1$. Evaluating at $z = 2$ gives $C = 1$. Finally, we compare coefficients of z^2 on both sides of (17): the coefficient on the left hand side is 0, while the coefficient on the right hand side is $A + C$. Thus $A + C = 0$, so $A = -C = -1$. Thus

$$\int_\gamma \frac{f(z)dz}{(z-1)^2(z-2)} = -\int_\gamma \frac{f(z)dz}{z-1} - \int_\gamma \frac{f(z)dz}{(z-1)^2} + \int_\gamma \frac{f(z)dz}{z-2}.$$

Noting that both 1 and 2 lie in the interior of γ and using Cauchy's Integral Formula for the first and last term and Cauchy's Integral Formula for the first derivative for the middle term, we get

$$\int_\gamma \frac{f(z)dz}{(z-1)^2(z-2)} = 2\pi i(-f(1) - (1!)f'(1) + f(2)) = 2\pi i(f(2) - f(1) - f'(1)).$$

This gives a good idea of the sort of integrals that we can compute using the present tools. For an integral that we'd like to compute but can't yet, consider $\int_\gamma \frac{e^z dz}{\sin z}$ for a simple closed curve γ. The roots of $\sin z$ are at $n\pi$ for $n \in \mathbb{Z}$, so if we are so fortunate that none of these points lie on the interior of γ, the integral is 0. In general, if the roots enclosed are $r_1, \ldots, r_n$, then the sort of dissection argument we've seen before shows that the integral is equal to the sum of integrals around small circles enclosing each of the roots r_1. So for instance taking γ to be the circle of radius 1 centered at π is representative of the general case. A little thought suggests writing the integrand as $\frac{e^z / \frac{\sin z}{z-\pi}}{z-\pi}$. Now the function $\frac{\sin z}{z-\pi}$ does not look holomorphic at π: strictly speaking, it is not even defined there. However, we have

$$\lim_{z\to\pi} \frac{\sin z}{z-\pi} = \lim_{z\to\pi} \frac{\sin z - \sin \pi}{z - \pi} = \frac{d}{dz} \sin z|_{z=\pi} = \cos\pi = -1.$$

In other words, if we extend $\frac{\sin z}{z-\pi}$ to take the value -1 at π then at least it is continuous there. If we *assume* that the function is holomorphic, then Cauchy's Integral Formula applies to give an integral of $2\pi i(-e^\pi)$. How do we justify this? Thinking back to the real variable case, we remember that the sine function is given by a convergent power series expansion around every point, so in particular about π:

$$\sin x = \sin(\pi) + \sin'(\pi)(x-\pi) + \sin''(\pi)/2(x-\pi)^2 + \ldots.$$

Since $\sin(\pi) = 0$, every term in this expansion is divisible by $(x - \pi)$. So therefore we can divide by $(x - \pi)$ and still get a power series that converges for all real x:

$$\frac{\sin x}{x-\pi} = \sin'(\pi) + \sin''(\pi)/2(x-\pi) + \sin'''(\pi)/6!(x-\pi)^2 + \ldots.$$

Any such function is infinitely differentiable. So in other words, if we can extend the theory of power series from a real variable to a complex variable, we can justify the above calculation. We proceed to do this shortly.

6. Series methods

6.1. **Convergence, absolute convergence and uniform convergence.** For a sequence $\{a_n\}_{n=1}^\infty$ of complex numbers, we say that the infinite series $\sum_{n=1}^\infty a_n$ converges to $S \in \mathbb{C}$ if the sequence of partial sums $S_n = a_1 + \ldots + a_n$ converges to S. As usual for convergence of complex series, this means precisely that the real part of S_n converges to the real part of S and the imaginary part of S_n converges to the imaginary part of S. Since Cauchy sequences in $\mathbb{C}$ converge, we get the usual Cauchy criterion for convergence of $\sum_{n=1}^\infty a_n$: namely, that this series converges iff for all $\epsilon > 0$ there is $N \in \mathbb{N}$ such that for all $k \geq N$ and $m \geq 0$ we have $|\sum_{n=k}^{k+m} a_n| < \epsilon$.

A series $\sum_{n=1}^{\infty} a_n$ is **absolutely convergent** if the associated sequence $\sum_{n=1}^{\infty} |a_n|$ of non-negative real numbers converges; recall that the latter holds iff the sequence of partial sums $|a_1| + \ldots + |a_n|$ is bounded.

Proposition 6.1. *An absolutely convergent complex series is convergent.*

Proof. There are two standard proofs of the corresponding fact for real numbers, and one of them carries over verbatim to complex series. Namely, we apply the Cauchy criterion twice: if $\sum_n a_n$ is absolutely convergent, then for all $\epsilon > 0$ there is $N \in N$ such that for $k \geq N$ and $m \geq 0$ we have $\sum_{n=k}^{k+m} |a_n| < \epsilon$, but then

$$|\sum_{n=k}^{k+n} a_n| \leq \sum_{n=k}^{k+n} |a_n| < \epsilon,$$

so by the Cauchy criterion, the series $\sum_n a_n$ converges. □

Example 6.2. *For $s \in \mathbb{C}$, we define $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$. Writing $s = \sigma + it$, we have $n^s = n^{\sigma+it} = n^\sigma n^{it} = n^\sigma (e^{\log n})^{it} = n^\sigma e^{i \log n t}$, so*

$$|n^s| = n^\sigma.$$

Since for real $s > 1$ the series $\sum_{n=1}^{\infty} \frac{1}{n^s}$ converges, we get that for $\Re(s) > 1$ the series defining $\zeta(s)$ is absolutely convergent. This function is – no kidding! – the most important function in all of mathematics. It is called the ***Riemann zeta function****.*

Let $A \subset \mathbb{C}$, and let $\{f_n : A \to \mathbb{C}\}_{n=1}^{\infty}$ be a sequence of functions. We say that f_n converges **pointwise** to $f : A \to \mathbb{C}$ if for all $a \in A$ we have $f_n(a) \to f(a)$. We say that f_n converges **uniformly on A** to $f : A \to \mathbb{C}$ if for all $\epsilon < 0$, there is $N \in \mathbb{N}$ such that for all $n > N$ and all $a \in A$ we have $|f_n(a) - f(a)| < \epsilon$. Now we have all of the folllowing $\mathbb{C}$ analogues of results over $\mathbb{R}$, with identical proofs.

Theorem 6.3. *Let $\{f_n : A \to \mathbb{C}\}$ and let $f : A \to \mathbb{C}$. If f_n converges uniformly to f on A and each f_n is continuous, then so is f.*

Theorem 6.4. *(Weierstrass M-Test)*
Let $\{f_n : A \to \mathbb{C}\}_{n=1}^{\infty}$ be a sequence of functions. For each n, let

$$||f_n|| = \sup\{|f_n(a)| \mid a \in A\} \in [0, \infty].$$

If $\sum_{n=1}^{\infty} ||f_n|| < \infty$, then the series $\sum_{n=1}^{\infty} f_n$ converges uniformly on A.

The following is a very close analogue of the fact that uniform convergence allows us to interchange limits with integrals.

Theorem 6.5. *Let $A \subset \mathbb{C}$, and let $\{f_n : A \to \mathbb{C}\}_{n=1}^{\infty}$ be a sequence of continuous functions that converge to $f : A \to \mathbb{C}$ uniformly on C. Let $\gamma : [a, b] \to A$ be any path. Then*

$$\lim_{n\to\infty} \int_\gamma f_n = \int_\gamma f.$$

Proof. It is equivalent to show that $\lim_{n\to\infty} \int_\gamma (f_n - f) \to 0$. But this follows almost immediately from the ML-Inequality: let L be the length of path. For $\epsilon > 0$, there is $N \in \mathbb{N}$ such that for all $n > N$ we have $|f_n(z) - f(z)| < \frac{\epsilon}{L}$ for all $z \in A$, and thus

$$|\int_\gamma f_n - f| \leq \frac{\epsilon}{L} L = \epsilon.$$

□

Theorem 6.6. *Let $U \subset \mathbb{C}$ be a domain, and let $\{f_n : U \to \mathbb{C}\}_{n=1}^{\infty}$ be a sequence. We assume:*
(i) Each f_n is holomorphic on U,
(ii) There is $f : U \to \mathbb{C}$ such that f_n converges to f pointwise on U,
(iii) The sequence f_n' converges uniformly on U to some holomorphic function $g : U \to \mathbb{C}$.
Then f is holomorphic on U and $f' = g$. In other words, $(\lim_{n\to\infty} f_n)' = \lim_{n\to\infty} f_n'$.

Proof. Since the conclusions are "local" on U it is no loss of generality to assume that U is an open disk, hence simply connected. Fix $a \in U$, and define $G : U \to \mathbb{C}$ by

$$G(z) := \int_a^z g.$$

The integral is independent of path because g is holomorphic and U is simply connected. Then

$$G(z) = \int_a^z g = \int_a^z \lim_{n\to\infty} f_n' = \lim_{n\to\infty} \int_a^z f_n' = \lim_{n\to\infty} f_n(z) - f_n(a) = f(z) - f(a).$$

As we know, $G(z)$ is an antiderivative of g, so

$$G'(z) = g(z) = f'(z).$$

□

Theorem 6.7. *Let $U \subset \mathbb{C}$ be a domain, and let $\{f_n : U \to \mathbb{C}\}_{n=1}^{\infty}$ be a sequence of holomorphic functions.*
a) If $f_n \to f$ uniformly on U, then f is holomorphic.
b) In fact the conclusion follows under the weaker assumption of "local uniform convergence": for all $a \in U$ there is $R > 0$ such that $B_a(R) \subset U$ and $f_n \to f$ uniformly on $B_a(R)$.

Proof. Since part b) implies part a), we will just prove part b). Let $\gamma : [a,b] \to B_a(R)$ be any simple closed curve. Since $B_a(R)$ is simply connected, by Cauchy's Integral Theorem we have $\int_\gamma f_n = 0$ for all $n \in \mathbb{Z}^+$. Since f_n converges to f uniformly on $B_a(R)$, by Theorem 6.5 we have

$$\int_\gamma f = \lim_{n\to\infty} \int_\gamma f_n = \lim_{n\to\infty} 0 = 0.$$

By Morera's Theorem, f is holomorphic on U. □

Example 6.8. *Let $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$. Earlier we showed that this series is absolutely convergent for all $\sigma = \Re(s) > 1$. In fact, for any fixed $p > 1$, the convergence is uniform on $\{s \in \mathbb{C} \mid \sigma = \Re(s) \geq p\}$ by the Weierstrass M-test:*

$$\sum_{n=1}^{\infty} |\frac{1}{n^s}| = \sum_{n=1}^{\infty} \frac{1}{n^\sigma} \leq \sum_{n=1}^{\infty} \frac{1}{n^p} < \infty.$$

This condition implies that for each fixed s with $\sigma = \Re(s) > 1$, the convergence is uniform on some open disk centered at s, so by the previous result the Riemann zeta function is holomorphic on $\Re(s) > 1$.

6.2. **Complex power series.** Let $c \in \mathbb{C}$. A **power series centered at a** is a series $\sum_{n=0}^{\infty} a_n(z-c)^n$, where $\{a_n\}_{n=0}^{\infty}$ is a sequence of complex numbers. As for real power series, we are interested in the set of values on which the power series converges.

Example 6.9. *Consider the geometric series $\sum_{n=0}^{\infty} z^n$. The discussion is very similar to the real case. At $z = 1$ we get $\sum_{n=0}^{\infty} 1$, which diverges. For $z \neq 1$ we have $\sum_{n=0}^{\infty} z^n = \lim_{N\to\infty} \sum_{n=0}^{N} z^n = \lim_{N\to\infty} \frac{1-z^{N+1}}{1-z}$. When $|z| < 1$, we have $z^{N+1} \to 0$, hence the series converges to $\frac{1}{1-z}$. For $|z| > 1$ we have $z^{N+1} \to \infty$, so the series diverges. The only tricky case is what happens when $|z| = 1$. I will you the answer, though it is more complicated than we need: if z is a root of unity (not equal to 1), then z^{N+1} cycles through finitely many (but more than one) different values, so the series* *diverges. Otherwise the sequence $\{z^{N+1}\}$ is* ***dense*** *in the unit circle: this means that given any $0 \leq \theta_1 < \theta_2 \leq 2\pi$, there are infinitely many $N \in \mathbb{Z}^+$ such that the argument of z^{N+1} lies in $[\theta_1, \theta_2]$. Again this means that the series diverges.*

So the answer is that the geometric series converges precisely when $|z| < 1$, so the set of convergence is the open unit disk. In a very reasonable sense the ***radius of convergence*** *ifs $R = 1$. We also get the hint though that trying to figure out what happens* on the boundary circle $R = 1$ *is more complicated than in the real case. That is a good moral.*

Theorem 6.10. *Let $\sum_{n=0}^{\infty} a_n(z-a)^n$ be a complex power series.*
a) There is $R \in [0, \infty]$, called the ***radius of convergence****, such that:*
(i) If $R = 0$ then the series converges only at $z = a$.
(ii) If $R = \infty$ then the series converges for all $z \in \mathbb{C}$.
(iii) The series converges absolutely for all z with $|z-a| < R$ and the series diverges for all z with $|z-a| > R$.
b) If $R > 0$, then for all $0 < r < R$, the series converges uniformly on the closed disk $\overline{B}_a(r)$.
c) If the Ratio Test limit $\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$ exists in $[0, \infty]$, then $R = \frac{1}{\rho}$.
d) (Cauchy-Hadamard) Let $\theta = \limsup_n |a_n|^{\frac{1}{n}}$.[12] *Then $R = \frac{1}{\theta}$.*

Proof. First observe that just by making the change of variables $w = z - a$ we may assume that $a = 0$. Let

$$\mathcal{S} := \{r \in [0, \infty) \mid \sum_{n=0}^{\infty} a_n z^n \text{ converges for some } z \text{ with } |z| = r\},$$

and let $R = \sup \mathcal{S} \in [0, \infty]$. We will show that this is the R in the statement of the theorem. Here is the key CLAIM: let $z_1, z_2 \in \mathbb{C}$ with $|z_2| < |z_1|$. If $\sum_n a_n z_1^n$ converges, then $\sum_n a_n z_2^n$ is absolutely convergent. Indeed, since $\sum_n a_n z_1^n$ converges, the sequence of terms converges to 0 and thus is bounded: there is $M > 0$ such that $|a_n z_1^n| \leq M$ for all n. Then

$$r := |\frac{z_2}{z_1}| < 1,$$

and we have

$$|a_n z_2^n| = |\frac{z_2}{z_1}|^n (|a_n z_1^n|) \leq |\frac{z_2}{z_1}|^n M \leq M r^n,$$

so by comparison to a convergent geometric series we get

$$\sum_n |a_n z_2^n| \leq \sum_n M r^n < \infty,$$

establishing the claim.
a) If $R = 0$, then the series only converges at $z = 0$, while if $R = \infty$ then for all $N \in \mathbb{Z}^+$ there is some $z \in \mathbb{C}$ with $|z| \geq N$ such that $\sum_n a_n z^n$ converges, and then the claim implies that the series converges absolutely for all z with $|z| \leq N$. Since N was arbitrary, this means that the series converges absolutely for all $z \in \mathbb{C}$. Finally, suppose $R \in (0, \infty)$. If $|z| > R$, then $\sum_n a_n z^n$ must diverge: otherwise $|z| \in \mathcal{S}$ so $|z| \leq \sup \mathcal{S} = R$. If $|z| < R$ then $|z|$ is smaller than the least upper bound for $\mathcal{S}$ so is not an upper bound for $\mathcal{S}$: there is $r \in \mathcal{S}$ with $|z| < r$. This means that there is $w \in \mathbb{C}$ with $|w| = r$ and such that $\sum_n a_n w^n$ converges, and then by the claim the series $\sum_n a_n z^n$ is absolutely convergent.
b) If $0 < r < R$, then by part a) we know that for all z with $|z| = r$ the series $\sum_n a_n z^n$ is absolutely convergent. So if $|w| \leq r$ then

$$\sum_n |a_n w^n| = \sum_n |a_n||w|^n \leq \sum_n |a_n| r^n < \infty,$$

[12]This is the largest limit of a subsequence of $|a_n|^{\frac{1}{n}}$.

so the convergence is uniform on $\overline{B}_0(r)$ by the Weierstrass M-Test.
c) This follows from the Ratio Test for absolute convergence of complex series in the usual manner.
d) ... □

Note that the only weak point in the above theorem is that we do not say anything about the convergence on the boundary circle $C_a(R)$. As alluded to above, the behavior here can be quite complicated, but these complications will not trouble us. Motivated by this, when $R > 0$ we define the **domain of convergence** of a power series to be the *open* disk $D_a(R)$ (which is $\mathbb{C}$ when $R = \infty$). As above, this is a minor lie in the sense that the series may converge at some of the points on the boundary circle, but it as the advantage that it makes the domain of convergence an actual *domain*, i.e., a connected open subset.

Theorem 6.11. *Let $\sum_{n=0}^{\infty} a_n(z-a)^n$ be a power series with radius of convergence $R \in (0, \infty]$. Consider the function*

$$f : B_a(R) \to \mathbb{C}, z \mapsto f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n.$$

a) The power series $\sum_{n=1}^{\infty} na_n(z-a)^{n-1}$ also has radius convergence R, and for all $z \in B_a(R)$.
b) The function f is holomorphic on $B_a(R)$ and for all $z \in B_a(R)$ we have

$$f'(z) = \sum_{n=1}^{\infty} na_n(z-a)^{n-1}.$$

c) It follows – without using Cauchy's Integral Formula for Derivatives! – that f is infinitely differentiable, and for all $n \in \mathbb{N}$, we have

$$a_n = \frac{f^{(n)}(a)}{n!}.$$

Proof. The argument is the same as its real variable analogue: see e.g. [C-HC, Thm. 13.10]. □

Example 6.12. *Consider the series $E(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$. The Ratio Test limit is*

$$\rho = \lim_{n\to\infty} \frac{1/(n+1)!}{1/n!} = \lim_{n\to\infty} \frac{1}{n+1} = 0,$$

so $R = \frac{1}{\rho} = \infty$. That is, the series converges converges for all $z \in \mathbb{C}$. By Theorem 6.11, $E(z)$ is an entire function. Moreover, we are allowed to differentiate termwise, so

$$E'(z) = \sum_{n=1}^{\infty} \frac{n}{n!} z^{n-1} = \sum_{n=0}^{\infty} \frac{z^n}{n!} = E(z).$$

Moreover, we have $E(0) = 1$. Well, I won't slow roll you – of course we want to say that $E(z)$ is just the exponential function e^z. Morally this comes from a uniqueness theorem for differential equations, but we can use the same sneaky trick from the proof of Theorem 4.30: put $g(z) = E(z)e^{-z}$. We have

$$g'(z) = E'(z)e^{-z} + E(z)(-e^{-z}) = E(z)e^{-z} - E(z)e^{-z} = 0.$$

So $g' \equiv 0$ on $\mathbb{C}$ and g is constantly equal to $g(0) = \frac{1}{1} = 1$. Thus $E(z) = e^z$ for all $z \in \mathbb{C}$.

Example 6.13. *Let $\{a_n\}$ be any* real *sequence and $a \in \mathbb{R}$. We can then consider the real power series*

$$\sum_{n=0}^{\infty} a_n(x-a)^n$$

alongside the complex power series

$$\sum_{n=0}^{\infty} a_n(z-a)^n.$$

Both power series have a radius of convergence, and it is not hard to see that the radii are the same. (E.g. the Cauchy-Hadamard formula is valid, with exactly the same proof, for real power series.) In particular, any real power series with an infinite radius of convergence defines an entire function on $\mathbb{C}$. Excepting only the previous example, our two favorite ones are

$$C(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$$

and

$$S(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}.$$

Again, from a differential equations perspective, the facts

$$C'' = -C,\ C(0) = 1,\ C'(0) = 0$$

and

$$S'' = -S,\ S(0) = 0,\ S'(0) = 1$$

strongly suggest that $C(z) = \cos z$ and $S(z) = \sin z$ for all $z \in \mathbb{C}$. We can prove this by series manipulatons:

$$e^{iz} = E(iz) = \ldots = C(z) + iS(z),$$

so

$$e^{-iz} = C(-z) + iS(-z) = C(z) - iS(z),$$

and thus we can solve for C and S:

$$C(z) = \frac{e^{iz} + e^{-iz}}{2},\ S(z) = \frac{e^{iz} - e^{-iz}}{2}.$$

6.3. **Analytic functions.** Let $U \subset \mathbb{C}$ be a domain, and let $f : U \to \mathbb{C}$. We say that f is **analytic** if for each $a \in U$ there is $r > 0$ with $B_a(r) \subset U$ and a power series $\sum_{n=0}^{\infty} a_n(z-a)^n$ with radius of convergence $R \geq r$ such that

$$\forall z \in B_a(r),\ f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n.$$

In fewer words, a function is analytic if in a small disk around every point in its domain it is given by a convergent power series. It follows from Theorem 6.11 that f is then infinitely differentiable and for all $n \in \mathbb{N}$ we have $a_n = \frac{f^{(n)}(a)}{n!}(z-a)^n$ and thus

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n,$$

i.e., f is equal to its **Taylor**[13] **series expansion** in (at least) some small open disk centered at a.

In particular an analytic function is holomorphic. This appears to be saying very little. Notice that the definition of an analytic function carries over verbatim to real functions $f : \mathbb{R} \to \mathbb{R}$, and the real analogue would be the statement that a function that is given by a convergent power series must be once differentiable. But that's an almost silly thing to say, since an analytic function is infinitely differentiable *and* has a convergent power series *and* is given by that power series,

[13]Brook Taylor, 1685-1731, was a British mathematician, known for the series and theorem that bear his name.

whereas a once differentiable real function need not even be twice differentiable. As we saw though, in the realm of complex variables, a holomorphic function must be infinitely differentiable. This perhaps gives us some hope that the following startling result might be true.

Theorem 6.14. *Let $U \subset \mathbb{C}$ be a domain, and let $f : U \to \mathbb{C}$ be holomorphic. Then f is analytic. More precisely: if $f : B_a(R) \to \mathbb{C}$ is holomorphic, for all $n \in \mathbb{N}$ let*

$$a_n := \frac{1}{2\pi i} \int_{C_a(R)} \frac{f(w)dw}{(w-a)^{n+1}}.$$

Then the power series

$$P(z) := \sum_{n=0}^{\infty} a_n (z-z)^n$$

has radius of convergence at least R, and for all $z \in B_a(R)$ we have

$$f(z) = P(z).$$

Proof. Step 1: We treat the special case $a = 0$. For $z \in B_0(R)$, let $r := \frac{|z|+R}{2}$, so $|z| < r < R$. By Cauchy's Integral Formula we have

$$f(z) = \frac{1}{2\pi i} \int_{C_0(r)} \frac{g(w)dw}{w-z}.$$

Now we have

$$\frac{1}{w-z} = \frac{1}{w} \frac{1}{1-\frac{z}{w}} = \frac{1}{w} \sum_{n=0}^{\infty} \left(\frac{z}{w}\right)^n.$$

For $w \in C_0(r)$ we have $|\frac{z}{w}| = \frac{|z|}{r} < 1$, so

$$|\frac{1}{w}| \sum_{n=0}^{\infty} \left(|\frac{z}{w}|\right)^n = \frac{1}{r} \sum_{n=0}^{\infty} (\frac{|z|}{r})^n < \infty,$$

so the convergence is uniform for $w \in C_0(r)$ by the Weierstrass M-Test. It follows that

$$f(z) = \frac{1}{2\pi i} \int_{C_0(r)} \frac{f(w)dw}{w-z} = \frac{1}{2\pi i} \int_{C_0(r)} f(w) \frac{1}{w} \sum_{n=0}^{\infty} (z/w)^n dw = \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{C_0(r)} \frac{f(w)dw}{w^{k+1}} \right) z^n$$
$$= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{C_0(R)} \frac{f(w)dw}{w^{k+1}} \right) z^n,$$

where in the last step we used the fact in Cauchy's Integral Theorem we can integrate around any simple closed curve, positively oriented, that encloses 0.
Step 2: In general, we apply Step 1 to $g(z) := f(z+a)$. □

Exercise 6.1. *Use Theorem 6.14 to (re)prove Cauchy's Integral Formula for Derivatives. (Hint: every power series with a positive radius of convergence is its own Taylor series.)*

The fact that every holomorphic function is analytic is truly remarkable, and is the third of the four main results in our course. Actually thought what we proved is even stronger than that in the following way: *a priori* a function is analytic if it is given in some *small* disk around every point in the domain by a convergent power series. But the above result shows that the radius of convergene is not so small: if we know *a priori* that f is holomorphic on a set U, then for $a \in U$ the radius of convergene is at least as large as the supremum of all $R > 0$ such that the disk $B_a(R)$ is contained in U. (It may in fact be larger; this is related to the important topic of **analytic continuation** that we will unfortunately not touch upon much here.) For instance:

Corollary 6.15. *If $f : \mathbb{C} \to \mathbb{C}$ is entire, then for all $a \in \mathbb{C}$, the Taylor series expansion converges for all $z \in \mathbb{C}$.*

Example 6.16. *Let $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \frac{1}{1+x^2}$. It can be shown that f is **real analytic** – for every $a \in \mathbb{R}$, the Taylor series expansion at a has positive radius of convergence and is equal to f. As we saw above, for a function $f : \mathbb{C} \to \mathbb{C}$ this would force the radius of convergence to be ∞ at every point. But that is not the case here. We have*

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}$$

and the Ratio Test shows that the radius convergence is 1. But at $x = \pm 1$, there is nothing so terribly wrong going on: in fact the power series are still (nonabsolutely) convergent at these points by the Alternating Series Test. And yet it is undeniably true tha the series diverges for all x with $|x| > 1$: indeed, the nth term approaches infinity in absolute value. Why?!?

Well, consider the complex function $f(z) = \frac{1}{1+z^2}$. It is not *entire: it is not defined at the roots of $1 + z^2$, i.e., at ± 1. Thus $f : \mathbb{C} \setminus \{i, -i\} \to \mathbb{C}$ is holomorphic. Applying Theorem 6.14 we find that for any $a \in \mathbb{C}$, the radius of convergence of the Taylor series at a is at least as large as*

$$\min(|a - i|, |a + i|).$$

(It turns out that this is exactly the radius of convergence: if it were any larger, then f would extend to a holomorphic function defined at either i or $-i$, but in fact f does not even extend continuously to these points. But this discussion will make much more sense once we establish the Identity Theorem.) Taking $a = 0$ we find that the radius of convergence is at least 1 and is not any larger because of the singularities at $\pm i$, which are each one unit away from 0. The behavior of real power series is completely governed by complex analysis!

6.4. **The Identity Theorem.** Let $P(z)$ be a nonzero polynomial, and let $a \in \mathbb{C}$ be such that $P(a) = 0$. We have the notion of the **multiplicity** of P at a. It can be expressed in several ways.

Exercise 6.2. *Let $P(z)$ be a nonzero polynomial, and let $a \in \mathbb{C}$.*
a) Show: there is a unique $m_1 \in \mathbb{N}$ such that $(z - a)^{m_1}$ divides $P(z)$ but $(z - a)^{m_1+1}$ does not.
b) Show: there is a unique $m_2 \in \mathbb{N}$ such that $P^{(k)}(a) = 0$ for all $0 \le k \le m_2$ but $P^{(m_2+1)}(a) \ne 0$.
c) Show: $m_1 = m_2$.

If $f(z) = \sum_n a_n(z-a)^n$ is a convergent power series that is not identically 0, then one has a similar notion of the multiplicity of f at a: it is the largest $n \in \mathbb{N}$ such that $a_n = 0$. Recall that $a_n = \frac{f^{(n)}}{n!}$, so this is the same as the second definition of multiplicity above. As in the above exercise, one can check that it is also the largest n such that one can factor $f(z)$ as $(z - a)^n$ times some other power series in $z - a$.

Using "holomorphic = analytic", we easily deduce the following result.

Proposition 6.17. *Let $U \subset \mathbb{C}$ be a domain, let $f : U \to \mathbb{C}$ be holomorphic, and let $a \in \mathbb{C}$. Then exactly one of the following holds:*
(i) There is $R > 0$ such that $B_a(R) \subset U$ and $f \equiv 0$ on $B_a(R)$.
*(ii) There is $m \in \mathbb{N}$ and a holomorphic function $g : U \to \mathbb{C}$ such that $f(z) = (z - a)^m g(z)$ and $g(a) \ne 0$. In this latter case, f has an **isolated zero** at a: namely, there is $0 < r \le R$ such that if $0 < |z| < r$ we have $f(z) \ne 0$.*

Proof. Choose $R > 0$ such that $B_a(R) \subset U$. Then $f(z) = \sum_n a_n(z-a)^n$ is given by a power series with radius of convergence at least R. If $a_n = 0$ for all n, then clearly $f \equiv 0$ on $B_a(R)$ and the first option holds. Otherwise there is a least natural number m such that $a_m \neq 0$, and thus

$$f(z) = \sum_{n=m}^{\infty} a_n (z-a)^n = (z-a)^m \sum_{n=0}^{\infty} a_n (z-a)^{n-m}.$$

If we put $g(z) = \sum_{n=0}^{\infty} a_n (z-a)^{n-m}$, then g is a power series with the same radius of convergence as the Taylor series of f at a, hence g is also holomorphic on $B_a(R)$ and $g(a) = a_m \neq 0$. This proves everything except that the zero is isolated, and that follows from continuity of g: since $g(a) \neq 0$, for z sufficiently close to a we have $|g(z) - g(a)| \leq \frac{|g(a)|}{2}$ and thus g is nonzero on $B_a(r)$ for some $r > 0$. □

Theorem 6.18. *(Identity Theorem) Let $U \subset \mathbb{C}$ be a domain, and let $A \subset U$ be a subset such that A has an accumulation point $a \in U$. Let $f : U \to \mathbb{C}$ and $g : U \to \mathbb{C}$ be two holomorphic functions. If $f(a) = g(a)$ for all $a \in A$, then $f(z) = g(z)$ for all $z \in U$.*

Proof. Step 0: By taking $h := f - g$, we immediately reduce to showing: if a holomorphic function h is identically zero on A, then h is identically zero on U.
Step 1: Let V be the set of $a \in U$ such that there is $R > 0$ with $B_a(R) \subset U$ and such that h is identically zero on $B_a(R)$. Let W be the set of $a \in U$ such that there is $R > 0$ with $B_a(R) \subset U$ and such that $h(z) \neq 0$ for all $0 < |z| < R$. Certainly $V \cap W = \varnothing$; we claim that $V \cup W = U$. Indeed, if $a \in U \setminus V$ then either $f(a) \neq 0$, in which case as in the proof of Proposition 6.17 f is nonzero on some small open disk centered at a, or $f(a) = 0$ and Proposition 6.17 tells us that the zero is isolated. We claim that U and V are each open. As for U, this holds for the same reason that an open disk is an open set: around any point in an open disk there is a smaller open disk centered at that point contained in the original open disk. The argument for V is similar: a continuous function that is nonzero at a point is nonzero in some small open disk centered at that point. Because U is connected, we must have either ($V = U$ and $W = \varnothing$) or ($W = U$ and $V = \varnothing$).
Step 2: We argue that $a \in V$. If so, then $W \neq U$, so we must have $W = \varnothing$ and $W = V$, which in particular implies that h is identically 0 on U. To see this, we first observe that since a is an accumulation point of A, there is a sequence $\{a_n\}$ in A converging to a Since h is holomorphic, it is continuous, and thus

$$h(a) = h(\lim_{n\to\infty} a_n) = \lim_{n\to\infty} h(a_n) = \lim_{n\to\infty} 0 = 0.$$

But a being an accumulation point of A means precisely that the zero at a is not isolated, so by Proposition 6.17 the function h must be identically zero in some open disk $B_a(R)$: that is, $a \in V$, completing the proof. □

Just to appreciate the power of this result, we isolate two important special cases.

Corollary 6.19. *Let $U \subset \mathbb{C}$ be a domain, and let $f, g : U \to \mathbb{C}$ be two holomorphic functions. If f and g agree on either*
(i) a line segment (not a point!) inside U, or
(ii) any nonempty open subset $V \subset U$,
then $f = g$.

Proof. Neither a line segment nor an open subset has any isolated points. □

Corollary 6.20. *(Maximum Modulus Principle) Let $U \subset \mathbb{C}$ be a domain, and let $f : U \to \mathbb{C}$ be holomorphic. If $|f|$ attains a local maximum value on U, then f is constant.*

Proof. By Theorem 4.3, if f attains a local maximum at $a \in U$ then f is constant on $B_a(R)$ for some $R > 0$ such that $B_a(R) \subset U$: that is, f agrees with the constant function $g \equiv C$ on the nonempty open subset $B_a(R)$. By the previous corollary, $f = g$, i.e., $f \equiv C$ on U. □

6.5. **Laurent series.** We need a mild extension of the concept of a complex infinite series; given a "doubly infinite sequence" $a_\bullet : \mathbb{Z} \to \mathbb{C}$, we want to associate the "doubly infinite series"

$$\sum_{n\in\mathbb{Z}} a_n.$$

We define a doubly infinite series $\sum_{n\in\mathbb{Z}} a_n$ to be convergent if the infinite series $\sum_{n=0}^{\infty} a_n$ converges and the infinite series $\sum_{n=1}^{\infty} a_{-n}$ converges, in which case we take the sum to be

$$\sum_{n\in\mathbb{Z}} a_n = \sum_{n=0}^{\infty} a_n + \sum_{n=1}^{\infty} a_{-n}.$$

Similarly, if both $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=1}^{\infty} a_{-n}$ diverge to ∞ we say that $\sum_{n\in\mathbb{Z}} a_n$ diverges to ∞. We say that a doubly infinite series is absolutely convergent if both $\sum_{n=0}^{\infty} |a_n|$ and $\sum_{n=1}^{\infty} |a_{-n}|$ converge.

Exercise 6.3.
a) For a doubly infinite sequence $a_\bullet : \mathbb{Z} \to [0,\infty)$, show that the following are equivalent:
(i) The series $\sum_{n\in\mathbb{Z}} a_n$ is convergent.
(ii) There is $M \in [0,\infty)$ such that for all finite subsets $S \subset \mathbb{Z}$ we have $\sum_{n\in S} a_n \leq M$.
b) Under the equivalent conditions of part a), show that the sum of the series is the least upper bound of the set of all $M \in [0,\infty)$ such that $\sum_{n\in S} a_n \leq M$.
c) Adapt the result of part a) to give a criterion for absolute convergence of a doubly infinite series.

For $\{a_n\}_{n=-\infty}^{\infty}$ a double sequence and $a \in \mathbb{C}$ we associate the **Laurent**[14] **series**

$$\sum_{n=-\infty}^{\infty} a_n(z-a)^n.$$

Thus a Laurent series is a direct generalization of a power series; otherwise put, a power series is a Laurent series in which $a_n = 0$ for all $n < 0$.

Example 6.21. *a) Consider the function $e^{1/z} = \sum_{n=0}^{\infty}(1/z)^n/n! = \sum_{n=-\infty}^{0} \frac{z^n}{|n|!}$. Since the power series for e^z converges for all $z \in \mathbb{C}$, the given series converges for all $z \neq 0$. It does not make sense to plug in $z = 0$ directly, but we can ask if the limit exists as $z \to 0$. Taking $w = \frac{1}{z}$, this is the same as asking whether $\lim_{w\to\infty} e^w$ exists, and we know that it doesn't. So the domain of convergence is $\mathbb{C} \setminus \{0\}$.*
b) In general if we start with a power series $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$ with radius of convergence $R > 0$ and consider the Laurent series

$$g(z) := f(1/(z-a)+a) = \sum_{n=0}^{\infty} a_n(z-a)^{-n},$$

then the series will converge for all z with $|\frac{1}{z-a}| < R$, or for all z with $|z-a| > \frac{1}{R}$.

In general, given a Laurent series

$$\sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{\infty} a_{-n}(z-a)^{-n}$$

convergence means that both series converge. The first series is an ordinary power series, say with radius of convergence R_1. As for the second series, as we saw above, if the radius of convergence of

[14] Pierre Alphonse Laurent, 1813-1854, was a French mathematician and military officer. Like Taylor, he is best known for the series that bears his name.

$\sum_{n=1}^{\infty} a_n(z-a)^n$ is R_2, then the second series converges if $|z-a| > \frac{1}{R_2}$ and diverges if $|z-a| < \frac{1}{R_2}$. Thus we define the **domain of convergence** of the Laurent series to be

$$U = \{z \in \mathbb{C} \mid \frac{1}{R_2} < |z-a| < R_1\}.$$

Here we require $R_1 > 0$, since otherwise the series does not converge anywhere, except at $z = a$ when $a_n = 0$ for all $n < 0$. Similarly we require $R_2 > 0$. We do allow $R_2 = \infty$, in which case we interpret $\frac{1}{R_2}$ as 0.

For $0 \leq R_1 < R_2 \leq \infty$ and $a \in \mathbb{C}$ we put

$$A^\circ(a, R_1, R_2) \coloneqq \{z \in \mathbb{C} \mid R_1 < |z-a| < R_2,$$
$$A^\bullet(a, R_1, R_2) \coloneqq \{Z \in \mathbb{C} \mid R_1 \leq |z-a| \leq R_2.$$

The set U is called an **open annulus**: it is open and connected but not simply connected. As with power series it may or may not be the case that the Laurent series converges on the boundary of U, and as with power series we do not concern ourselves with this.

Exercise 6.4. *Let $\sum_{n\in\mathbb{Z}} a_n(z-a)^n$ be a Laurent series convering in the annulus $A(a, R_1, R_2)$. Show: for any r_1, r_2 with $R_1 < r_1 < r_2 < R_2$, the convergence is uniform on $A^\bullet(a, r_1, r_2)$.*

It follows immediately from the holomorphicity of power series that every Laurent series determines a holomorphic function on its domain of convergence U. And the converse is also true, a generalization of Theorem 6.14.

Theorem 6.22. *Let $f : A(a, R_1, R_2) \to \mathbb{C}$ be holomorphic. Fix r such that $R_1 < r < R_2$. For all $n \in \mathbb{Z}$, put*

$$a_n = \frac{1}{2\pi i} \int_{C(a,r)} \frac{f(w)dw}{(w-a)^{n+1}}.$$

Then for all $z \in A(a, R_1, R_2)$, the Laurent series $\sum_{n=-\infty}^{\infty} a_n(z-a)^n$ converges to $f(z)$.

Proof. This is a generalization of Theorem 6.14, and the proof is rather similar.
Step 1: We treat the special case $a = 0$. Fix r_1 and r_2 with

$$R_1 < r_1 < r_2 < R_2,$$

and let $z \in A(a_1, r_1, r_2)$. Now we use essentially the same contours as in the proof of Cauchy's Integral Formiula: let γ_2 begin with the straight line segment from r_2 to r_1, followed by the upper semicircular arc on the inner circle to $-r_1$, then the straight line segment from $-r_1$ to $-r_2$ and finally the upper semicircular arc to r_2. Let γ_1 be the contour beginning at r_2, taking the bottom semicircular arc to $-r_2$, then the straight line segment from $-r_2$ to $-r_1$ and finally the straight line segment from r_1 to r_2. Now consider

$$\int_{\gamma_2} \frac{f(w)dw}{w-z} - \int_{\gamma_1} \frac{f(w)dw}{w-z}.$$

Then on the one hand, integrating over $\gamma_2 - \gamma_1$ is the same as integrating over $C(0, r_2) - C(0, r_1)$, since the contributions over the two segments cancel out. On the other hand, if $\Im(z) > 0$ then $\frac{f(w)}{w-z}$ is holomorphic on and inside γ_1, so $\int_{\gamma_1} \frac{f(w)dw}{w-z} = 0$ by Cauchy's Integral Theorem, whereas $\int_{\gamma_2} \frac{f(w)dw}{w-z} = 2\pi i f(z)$ by Cauchy's Integral Formula. If $\Im(z) < 0$, then $\frac{f(w)}{w-z}$ the same holds with the roles of γ_1 and γ_2 reversed: $\int_{\gamma_2} \frac{f(w)dw}{w-z} = 0$ and $-\int_{\gamma_1} \frac{f(w)dw}{w-z} = 2\pi i f(z)$ (the minus sign is because γ_1 is negatively oriented). Thus either way we find

$$g(z) = \frac{1}{2\pi i} \int_{\gamma_2 - \gamma_1} \frac{g(w)dw}{w-z} = \frac{1}{2\pi i} \int_{C(0,r_2)} \frac{g(w)dw}{w-z} - \int_{C(0,r_1)} \frac{g(w)dw}{w-z}.$$

Now we use the geometric series expansions

$$\frac{1}{w-z} = \frac{1}{w}\frac{1}{1-z/w} = \frac{1}{w}\sum_{n=0}^{\infty}\left(\frac{z}{w}\right)^n,$$

$$\frac{1}{w-z} = \frac{-1}{z}\frac{1}{1-w/z} = \frac{-1}{z}\sum_{n=0}^{\infty}\left(\frac{w}{z}\right)^n.$$

We leave to the reader the task of using uniform convergence to justify the following interchange of sums and integrals (it is much as in the proof of Theorem 6.14): we get

$$\begin{aligned}
g(z) &= \frac{1}{2\pi i}\int_{C(0,r_2)} g(w)\frac{1}{w}\sum_{n=0}^{\infty}(z/w)^n - \left(-\int_{C(0,r_1)} g(w)\frac{1}{z}\sum_{n=0}^{\infty}(w/z)^n dw\right)\\
&= \frac{1}{2\pi i}\sum_{n=0}^{\infty}\left(\int_{C(0,r_2)}\frac{g(w)dw}{w^{n+1}}\right)z^n + \frac{1}{2\pi i}\sum_{n=0}^{\infty}\left(\int_{C(0,r_1}g(w)w^n dw\right)z^{-n-1}\\
&= \frac{1}{2\pi i}\left(\sum_{n=0}^{\infty}\left(\int_{C(0,r_2)}\frac{f(w)}{w^{n+1}}dw\right)z^n + \sum_{n\leq -1}\left(\int_{C(0,r_1)}\frac{f(w)}{w^{n+1}}dw\right)z^n\right)\\
&= \frac{1}{2\pi i}\sum_{n=-\infty}^{\infty}\left(\int_{C(0,r)}\frac{f(w)dw}{w^{n+1}}\right)z^n,
\end{aligned}$$

the last equality by Theorem 4.17. This gives part a).
Step 2: In general, we apply Step 1 to $g(z) := f(z+a)$. □

Corollary 6.23. *Given two Laurent series $\sum_{n\in\mathbb{Z}} a_n(z-a)^n$ and $\sum_{n\in\mathbb{Z}} b_n(z-a)^n$, if for some $0 < R_1 < R_2 \leq \infty$ we have that $\sum_{n\in\mathbb{Z}} a_n(z-a)^n = \sum_{n\in\mathbb{Z}} b_n(z-a)^n$ for all $z \in A(a, R_1, R_2)$, then $a_n = b_n$ for all $n \in \mathbb{Z}$.*

Corollary 6.24. *Let U be a domain, let $a \in U$, and suppose $f : U \setminus \{a\} \to \mathbb{C}$ is holomorphic. Let R be the distance from a to the boundary of U. Then f is given by a Laurent series expansion on the annulus $A(a, 0, R)$.*

The following result is simply the $n = -1$ case of Theorem 6.22. At the same time it leads to one of the most important results in complex analysis!

Corollary 6.25. *Let $f : A(a, R_1, R_2) \to \mathbb{C}$ be holomorphic with Laurent series $\sum_{n\in\mathbb{Z}} a_n(z-a)^n$. Let γ be any simple closed curve in $A(a, R_1, R_2)$ that encloses a. Then we have*

$$\int_{\gamma} f = 2\pi i a_{-1}.$$

7. Isolated singularities

Let $U \subset \mathbb{C}$ be a domain, and let $f : U \to \mathbb{C}$ be a holomorphic function. For $a \in \mathbb{C}$, we say that f has an **isolated singularity** at a if
(IS1) $a \notin U$ and
(IS2) There is $R > 0$ such that $B_a(R) \setminus \{a\} \subset U$.

In other words, if f has an isolated singularity at a then it is not defined at a but is defined and holomorphic at all other points of some open disk centered at a.

Exercise 7.1. *Suppose the holomorphic function $f : U \to \mathbb{C}$ has an isolated singularity at $a \in \mathbb{C}$. Show that $U \cup \{a\}$ is open.*

In particular then a is an accumulation point of U, so we can study the limiting behavior of f as we approach a. In fact, that is exactly what we want to do.

Notice that our definition does *not* imply that there is "anything bad" happening at a in any sense. Indeed, if $f : U \to \mathbb{C}$ is holomorphic and $a \in U$, we can define $g : U \setminus \{a\} \to \mathbb{C}$ just by restricting f to $U \setminus \{a\}$, and by definition g has an isolated singularity at a. This seems a bit silly – why did we take a out of the domain?!? – but it is not immediately clear when we are in this "silly situation." Our first order of business is to nail this down.

Proposition 7.1. *Let $g : U \to \mathbb{C}$ be holomorphic and have an isolated singularity at $a \in \mathbb{C} \setminus U$. The following are equivalent:*
(i) There is a holomorphic function $f : U \cup \{a\} \to \mathbb{C}$ such that $f|_U = g$.
(ii) The limit $\lim_{z\to a} g(z)$ exists in $\mathbb{C}$.
(iii) We have $\lim_{z\to a}(z-a)g(z) = 0$.

Proof. (i) $\implies$ (ii) follows immediately from the fact that differentiable functions are continuous.
(ii) $\implies$ (iii) If $\lim_{z\to a} g(z) = L \in \mathbb{C}$, then

$$\lim_{z\to a}(z-a)g(z) = (\lim_{z\to a}(z-a))(\lim_{z\to a} g(z)) = 0 \cdot L = 0.$$

(iii) $\implies$ (i): Consider the function $h : U \cup \{a\} \to \mathbb{C}$ defined by

$$h(z) = \begin{cases} (z-a)^2 g(z) & z \neq a \\ 0 & z = a \end{cases}.$$

Our hypothesis (iii) is precisely what we need to get that h is differentiable at a and has derivative 0: indeed

$$h'(a) = \lim_{z\to a} \frac{h(z)-h(a)}{z-a} = \lim_{z\to a} \frac{(z-a)^2 g(z)}{z-a} = \lim_{z\to a}(z-a)g(z) = 0.$$

Thus h is analytic, and its Taylor series expansion at a begins

$$h(z) = h(a) + h'(a)(z-a) + \sum_{n=2}^{\infty} a_n(z-a)^n = 0 + 0(z-a) + \sum_{n=2}^{\infty} a_n(z-a)^n = (z-a)^2 \sum_{n=0}^{\infty} a_{n+2}(z-a)^n.$$

This shows that $h(z)$ is equal to $(z-a)^2$ times a function f that is holomorphic at a, namely $f(z) = \sum_{n=0}^{\infty} a_{n+2}(z-a)^n$. Since for all $z \neq a$ we have $f(z) = \frac{h(z)}{(z-a)^2} = g(z)$, this shows that f is the desired function, namely it restricts to g on U and is holomorphic at a. ☐

A function that satisfies the equivalent conditions of Proposition 7.1 is said to have a **removable singularity at a**. (Recall that a function is said to have a removable discontinuity at a point if the limit exists at that point.) This is precisely the case where there is "nothing wrong as f approaches a." In every other case $\lim_{z\to a} f(z)$ fails to exist. A little thought shows that the "next level of pathology" ought to be that $\lim_{z\to a} f(z) = \infty$. This turns out to be correct, although a different definition is more immediately useful to work with. Namely, let $f : U \to \mathbb{C}$ have an isolated singularity at a. We say that f **has a pole at a** if $\lim_{z\to a} f(z)$ does not exist in $\mathbb{C}$ but there is $m \in \mathbb{Z}^+$ such that

$$\lim_{z\to a}(z-a)^m f(z) \in \mathbb{C}.$$

An argument very similar to that of the proof (ii) $\implies$ (iii) in Proposition 7.1 shows that if

$$\lim_{z\to a}(z-a)^m f(z) \in \mathbb{C}$$

then for all integers $M > m$ we have

$$\lim_{z\to a}(z-a)^m f(z) = 0.$$

Thus if f has a pole at a then it is natural to consider the *least* m such that $\lim_{z\to a}(z-a)^m f(z) \in \mathbb{C}$; this is called the **order of the pole** of f at a. If the order is 1 we say that f has a **simple pole** at a. Here is a simple but important exercise.

Exercise 7.2. *Let $f(z) = \frac{P(z)}{Q(z)}$ be a rational function such that $P(z)$ and $Q(z)$ have no common roots. (Any rational function can be put in this form just by cancelling any common roots of the numerator and denominator; this is the analogue of putting a fraction in lowest terms.) Show: the isolated singularities of f are precisely the roots of $Q(z)$, these isolated singularities are all poles, and the order of the pole at a is the multiplicity of a as a root of $Q(z)$.*

Exercise 7.3. *Let $f, g : U \to \mathbb{C}$ be holomorphic functions having isolated singularities at $a \in \mathbb{C}\setminus U$.*
a) Show: $f + g : U \to \mathbb{C}$ and $fg : U \to \mathbb{C}$ also have isolated singularities at a.
b) Suppose g is not identically zero. Show that there is $R > 0$ such that $\frac{1}{g} : B_a(R) \setminus \{a\} \to \mathbb{C}$ is defined and holomorphic, with an isolated singularity at a.
c) Show: if the singularity at both f and g is removable, then so is the singularity of $f+g$ and fg.
d) Show: if f has a pole of order m_f at a and g has a pole of order m_g at a then fg has a pole of order $m_f + m_g$ at a.
e) Show: if f has a pole of order m_f at a, then $1/f$ has a removable singularity at a and the holomorphic extension of f to a has a zero of order m_f at a.
f) Suppose f has a pole of order m_f at a and g has a pole of order m_g at a. Put $M = \max(m_f, m_g)$. Show:

$$\lim_{z\to a}(z-a)^M((f(z)+g(z)) \in \mathbb{C}.$$

Deduce that $f + g$ has either a pole or a removable singularity at a and give examples to show that both are possible.

If a holomorphic function $f : U \to \mathbb{C}$ has an isolated singularity at a, let $R > 0$ be such that $B_a(R) \setminus \{a\} \subset U$. Then f has a Laurent series expansion on $B_a(R) \setminus \{a\}$:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-a)^n.$$

Now we observe:
• The singularity at a is removable iff $a_n = 0$ for all $n < 0$. Indeed, if $a_n = 0$ for all $n < 0$ then the Laurent series expansion is actually a power series expansion and shows that f is holomorphic at a. Conversely, if the singularity is removable then f extended to a admits a power series expansion at a which is then a Laurent series expansion in $B_a(R) \setminus \{a\}$ with no negative terms. By the uniqueness of Laurent series expansions, this means that the original Laurent series expansion must not have any negative terms.
• The singularity at a is a pole if $\{n \mid a_n \neq 0\}$ has a smallest element $N < 0$. In this case, $|N|$ is the order of the pole at a. Indeed, if the Laurent series expansion is of the form

$$\sum_{n=N}^{\infty} a_n(z-a)^n = \frac{a_N}{(z-a)^{|N|}} + \frac{a_{n+1}}{(z-a)^{|N|-1}} + \ldots$$

and multiplying by $(z-a)^{|N|}$ we get a power series, so we have a pole of order at most N. On the other hand, for any Laurent series of the form $\sum_{n=N}^{\infty} a_n(z-a)^n$ with $N < 0$ we have

$$\lim_{z\to a}\sum_{n=N}^{\infty} a_n(z-a)^n = \infty,$$

and thus if we multiplied by $(z-a)^m$ with $1 \leq m < |N|$, then we still have a Laurent series of the above form and thus the multiplicity of the pole must be exactly $|N|$. Conversely, if f has a pole order m at a, then $\lim_{z\to a}(z-a)^m f(z)$ exists, so $(z-a)^m f(z)$ has a removable singularity at a and is thus given by a power series expansion $\sum_{n=0}^{\infty} a_n(z-a)^n$, and then dividing through by $(z-a)^m$ gives

$$\sum_{n=0}^{\infty} a_n(z-a)^{n-m},$$

which is a Laurent series expansion with a smallest element that is at least $-m$. If the smallest element were non-negative then we would have a power series expansion and thus not a pole at a.

Proposition 7.2. *Let $f : U \to \mathbb{C}$ be a holomorphic function with an isolated singularity at a. Then the following are equivalent:*
(i) f has a pole at a.
(ii) We have $\lim_{z\to a} f(z) = \infty$.

Proof. (i) $\Longrightarrow$ (ii): For $R > 0$ such that $B_a(R) \setminus \{a\} \subset U$, if the order of the pole at f is m, then f has a Laurent series expansion

$$f(z) = \sum_{n=-m}^{\infty} a_n(z-a)^n.$$

Let $g = \sum_{n=-m}^{-1} a_n(z-a)^n$ and $h = \sum_{n=0}^{\infty} a_n(z-a)^n$. Then g is a rational function whose denominator has a root at a and h is holomorphic at a, so

$$\lim_{z\to a} f(z) = \lim_{z\to a} g(z) + \lim_{z\to a} h(z) = \infty + a_0 = \infty.$$

(ii) $\Longrightarrow$ (i): If $\lim_{z\to a} f(z) = \infty$ then $\lim_{z\to\infty} \frac{1}{f(z)} = 0$, so $\frac{1}{f}$ extends to a holomorphic function at a with an isolated zero at a. Thus we may write

$$\frac{1}{f(z)} = (z-a)^m g(z)$$

with g holomorphic at $g(a) \neq 0$, and so we can write

$$f = \frac{1}{(z-a)^m}\frac{1}{g(z)}.$$

Since $g(a) \neq 0$, $\frac{1}{g(z)}$ is holomorphic and nonvanishing at a, and it is now clear that f has a pole of order m at a. $\square$

Proposition 7.3. *Suppose the holomorphic function $f : U \to \mathbb{C}$ has an isolated singularity at a and $B_a(R) \setminus \{a\} \subset U$. Then the following are equivalent:*
(i) For no $m \in \mathbb{Z}^+$ does $\lim_{z\to a}(z-a)^m f(z)$ exist.
(ii) In the Laurent series expansion $\sum_{n\in\mathbb{Z}} a_n(z-a)^n$ of f on $B_a(R)\setminus\{a\}$, there are infinitely many negative n such that $a_n \neq 0$.
When these equivalent conditions are satisfied, we say that f has an ***essential singularity at a****.*

Proof. It follows from the previous discussion that if we have either a removable singularity or a pole that neither (i) nor (ii) holds and that if have neither a removable singularity nor a pole then both (i) and (ii) hold. $\square$

Example 7.4. *One way of producing essential singularities is just to start with an entire function $f : \mathbb{C} \to \mathbb{C}$ that is not a polynomial. Then in the power series expansion $\sum_{n=0}^{\infty} a_n z^n$ we have $a_n \neq 0$ for infinitely many n. Now consider*

$$g(z) := f(1/z) = \sum_{n=0}^{\infty} \frac{a_n}{z^n} = \sum_{n=-\infty}^{0} a_{-n} z^n.$$

Then g has an essential singularity at 0. Moreover, we have

$$\lim_{z \to 0} f(1/z) = \lim_{z \to \infty} f(z),$$

so the limiting behavior of the essential singularity in this case is equivalent to the limiting behavior of the entire function f at infinity. One's first guess might be that $\lim_{z \to 0} g(z) = \infty$ as with a pole, but this is not borne out by examples. For instance, take $f(z) = e^z$. Then if we approach ∞ through positive real numbers then we get ∞; if we aproach infinity through negative real numbers, then we get zero; if we approach infinity along the positive imaginary axis, then we get a function that is bounded but not convergent. So in this case the limit does not exist in rather dramatic fashion.

Exercise 7.4. *Let $\alpha \in \mathbb{C} \setminus \{0\}$.*
a) Show that for all $R > 0$ there is $z \in \mathbb{C}$ such that $|z| > R$ and $e^z = \alpha$.
(Suggestion: use the fact that $e^z = e^{z+2\pi i}$ for all $z \in \mathbb{C}$.
b) Deduce: for all $r > 0$ there is $z \in B_0(r) \setminus \{0\}$ such that $e^{1/z} = \alpha$.

The above exercise gives an example of an essential singularity at a such that in any deleted disk about a every value except 0 is assumed. Remarkably, this is not particular to the exponential function at all.

Theorem 7.5. *(Picard[15], 1879) Let $f : U \to \mathbb{C}$ have an essential singularity at a, and let $R > 0$ be such that $B_a(R) \setminus \{a\} \subset U$. Then $f(B_a(R) \setminus \{a\})$ is either all of $\mathbb{C}$ or consists of all but one point of $\mathbb{C}$.*

Corollary 7.6. *Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function that is not a polynomial. Then there is at most one $\alpha \in \mathbb{C}$ such that $f^{-1}(\{\alpha\}) = \{z \in \mathbb{C} \mid f(z) = \alpha\}$ is finite. In other words, a transcendental entire function takes on every complex value infinitely many times, with one possible exception.*

Proof. As above, we apply Picard's Theorem to $g(z) \coloneqq f(1/z)$. □

The proof of Theorem 7.5 is beyond the scope of this course. Well beyond – it is not usually proved in a first graduate course on complex analysis. However, a weaker statement is much easier to prove. First, we say that a subset $A \subset \mathbb{C}$ is **dense** in $\mathbb{C}$ if every point of $\mathbb{C}$ is an accumulation point of A. Equivalently, given any $w \in \mathbb{C}$ and $R > 0$, the disk $B_w(R)$ intersects A. (If you were expecting to hear $(B_w(R) \setminus \{w\} \cap A) \neq \varnothing$, then this stronger condition is equivalent here: $B_z(R) \setminus \{z\}$ contains an open disk around some other point w, so if A does not meet the *deleted* disk then it does not meet some other nondeleted disk.)

Theorem 7.7. *(Casorati[16]-Weierstrass) Let $f : U \to \mathbb{C}$ be holomorphic with an essential singularity at $a \in \mathbb{C}$. For any $R > 0$ such that $B_a(R) \setminus \{a\} \subset U$, the set $f(B_a(R) \setminus \{a\})$ is dense in $\mathbb{C}$.*

Proof. It is enough to show: if there is $w \in \mathbb{C}$ and $\epsilon > 0$ such that for all $z \in B_a(R) \setminus \{a\}$ we have

$$|f(z) - w| \geq \epsilon,$$

then f does not have an essential singularity at a. Since our assumption implies that $f(z) \neq w$ for any $z \in B_a(R) \setminus \{a\}$, we may define

$$g : B_a(R) \setminus \{a\} \to \mathbb{C}, z \mapsto \frac{1}{f(z) - w}.$$

[15] Charles Émile Picard, 1856-1941, was a French mathematician, whose work played an important role in the unification of large portions of algebra, geometry and complex analysis.

[16] Felice Casorati, 1835-1890, was a (male!) Italian mathematician.

Moreover, our assumption gives that g is bounded on $B_a(R) \setminus \{a\}$. Thus

$$0 = \lim_{z \to a} (z-a)g(z) = \lim_{z \to a} \frac{z-a}{f(z)-w}.$$

Taking reciprocals, we get

$$\lim_{z \to a} \frac{f(z)-w}{z-a} = \infty,$$

so the function $\frac{f(z)-w}{z-a}$ has a pole at a by Proposition 7.2, so only finitely many negative terms appear in its Laurent series expansion. This is not changed by multiplying by $z-a$ and adding w, so also $f(z)$ has only finitely many negative terms in its Laurent series expansion. Thus f does not have an essential singularity at a, completing the proof. □

8. The residue calculus

8.1. The Residue Theorem.

Let $f : U \to \mathbb{C}$ be holomorphic with an isolated singularity at a. As above, we have a Laurent series expansion

$$f(z) = \sum_{n \in \mathbb{Z}} a_n (z-a)^n.$$

We define the **residue of f at a** to be the coefficient a_{-1} of the Laurent series expansion. We denote it by $\mathrm{Res}(f; a)$.

Why so much ado about a single coefficient in the Laurent series expansion? In fact it is not hard to see that the a_{-1} coefficient is special. Namely, let γ be a simple closed curve, positively oriented, such that f is holomorphic on the interior of γ except possibly at a. For $n \in \mathbb{Z}$, consider

$$\int_\gamma a_n (z-a)^n dz.$$

If $n \neq -1$, then the integrand $f(z) = a_n(z-a)^n$ has a holomorphic antiderivative on U, namely

$$F(z) = \frac{a_n}{n+1}(z-a)^{n+1},$$

and thus the integral around the closed path γ is 0. This breaks down when $n = -1$; rather, in this case we get – by Cauchy's Integral Formula, but really by the direct calculation we used to establish it –

$$\int_\gamma \frac{a_{-1}}{z-a} = 2\pi i a_{-1}.$$

In fact one can show that the image of γ lies in a closed, bounded subannulus on which the convergence of the Laurent series is uniform, and this implies that we can interchange the series with the integral:

$$\int_\gamma f = \int_\gamma \sum_{n \in \mathbb{Z}} a_n (z-a)^n = \sum_{n \in \mathbb{Z}} \int_\gamma a_n (z-a)^n = 2\pi i a_{-1}.$$

We do not need to supply the details because we have seen this result before: Corollary 6.25. (This result was an immediate consequence of Theorem 6.22. The difficult part of this theorem was to show that any holomorphic function on an annulus is given by a Laurent series. Above we assumed that we had a convergent Laurent series expansion, and that makes things much easier.) As we have mentioned earlier in the course, this shows that $\frac{1}{z-a}$ has no holomorphic antiderivative on U, and the problem is again that of an absence of logarithms on a domain that is not simply connected (as U is not). Anyway, rewriting this result with our new terminology we get

$$\int_\gamma f = 2\pi i \operatorname{Res}(f; a).$$

In the above result we considered an integral that has only one isolated singularity on the interior of the curve. But using a familiar "dissection" argument we can quickly extend this result.

Well, first we want a preliminary result of a mostly topological nature.

Lemma 8.1. *Let γ be a simple closed curve in $\mathbb{C}$, and let C be the closed, bounded set consisting of γ and its interior. Let U be a domain containing C. Let $A \subset C$ be a subset without any accumulation points. Let $f : U \setminus A \to \mathbb{C}$ be holomorphic. Then:*
a) For all $a \in A$, the function f has an isolated singularity at A.
b) The set A is finite.

Proof. a) Let $a \in A$. Since A has no accumulation points, there is $R > 0$ such that $B_a(R) \subset U$ and $B_a(R) \cap A = \{a\}$. So f has an isolated singularity at a.
b) Seeking a contradiction, we suppose that A is infnite, and let $\{a_n\}_{n=1}^\infty$ is a sequence of distinct elements of A. Since C is closed and bounded, by Bolzano- Weierstrass there is a subsequence $\{a_{n_k}\}$ converges to some $a \in C$. Now we must have $a \in A$: otherwise, f is holomorphic at a and thus is defined in some open disk around a, but f is not defined at a_{n_k} for any k and thus at points arbitrarily close to a. So $a \in A$, but then a is an accumulation point of A: contradiction. $\square$

Theorem 8.2. *(Residue Theorem) Let γ be a simple closed curve in $\mathbb{C}$, positively oriented. LEt C be the closed, bounded set consisting of γ and its interior. Let U be a domain that contains γ and its interior. Let $A \subset C$ be a subset without any accumulation points, and let $f : U \setminus A \to \mathbb{C}$ be holomorphic. We suppose that no point of A lies on the image of γ. Then:*
a) By Lemma 8.1, A is finite, so we may write $A = \{a_1, \ldots, a_n\}$.
b) We have

$$\int_\gamma f = 2\pi i \sum_{k=1}^n \operatorname{Res}(f; a_k).$$

Proof. Step 1: Observe: when A consists of a single point, this is precisely Corollary 6.25.
Step 2: In the general case, for $1 \leq k \leq n$, let γ_k be a small circle centered at a_k and positively oriented. How small? Small enough to be contained in the interior of γ and small enough that none of the circles intersect each other. Then by Step 1 we have

$$\sum_{k=1}^n \int_{\gamma_k} f = 2\pi i \sum_{k=1}^n \operatorname{Res}(f; a_k).$$

Now, using auxiliary simple arcs we can subdivide the region lying interior to γ and exterior to all of the γ_k's into a finite union of simply connected domains D_i on which f is holomorphic, so by the Cauchy Integral Theorem the integral around each boundary ∂D_i is 0. However as we trace along all these boundaries in the positive direction we find that each auxiliary arc gets traversed once with each orientation, so these contributions to the integral cancel out. We also get γ and the circles $\gamma_1, \ldots, \gamma_n$ traversed with negative orientation. It follows that

$$\int_\gamma f + \int_{-\gamma_1} f + \ldots + \int_{-\gamma_n} f = 0,$$

or

$$\int_{\gamma} f = \sum_{k=1}^{n} \int_{\gamma_k} f = 2\pi i \sum_{k=1}^{n} \text{Res}(f; a_k) \qquad \square.$$

8.2. **How to compute residues.** The Residue Theorem is our single best weapon for computing integrals – both complex contour integrals and integrals along the real line that we evaluate using them. However, in order to properly wield it we need some further technique in computing residues. One would think that to compute $\text{Res}(f;a)$ we should just compute the Laurent series expansion of f at a. However, this is asking us to compute a_n for all $n \in \mathbb{Z}$ only to throw away everything but $n = -1$. In practice this is way too much work: there must be a better way! Here are some:

Theorem 8.3. *a) Suppose the function f has a pole of order $N \geq 1$ at a then*

$$\text{Res}(f;a) = \frac{1}{(N-1)!}((z-a)^N f(z))^{(N-1)}|_{z=a}. \tag{18}$$

b) Suppose f and g are holomorphic at a and g has a zero of multiplicity 1 at a. Then

$$\text{Res}(f/g;a) = \frac{f(a)}{g'(a)}. \tag{19}$$

c) Suppose f and g are holomorphic at a and g has a zero of multiplicity 2 at a. Then

$$\text{Res}(f/g;a) = \frac{2f'(a)}{g''(a)^2} - \frac{2}{3}\frac{f(a)g'''(a)}{g''(a)^2}. \tag{20}$$

Proof. a) By assumption, we have

$$f(z) = \sum_{n \geq -N} a_n (z-a)^n,$$

so

$$(z-a)^N f(z) = \sum_{n \geq 0} a_{n-N}(z-a)^n$$

is a power series expansion, so $(z-a)^N f(z)$ is holomorphic at a. (More fastidiously, it has a removable singularity at a. Okay: we remove it!) The coefficient a_{-1} that we want to compute corresponds to $n = N-1$, so by the usual Taylor series formula we take the $(N-1)$st derivative at a and divide by $(N-1)!$
b) If $f(a) = 0$, then f/g is holomorphic at a, so $\text{Res}(f/g;a) = 0 = \frac{f(a)}{g'(a)}$. So suppose $f(a) \neq 0$. Then f/g has a simple pole at a. Applying part a) with $N = 1$ we get

$$\text{Res}(f/g;a) = \frac{(z-a)f(z)}{g(z)}|_{z=a}.$$

Since g has multiplicity 1 at a, write $g(z) = (z-a)h(z)$ with h holomorphic and $h(a) \neq 0$. Then

$$\frac{(z-a)f(z)}{g(z)} = \frac{(z-a)f(z)}{(z-a)h(z)} = \frac{f(z)}{h(z)},$$

and evaluating at a we get

$$\text{Res}(f/g;a) = \frac{f(a)}{h(a)}.$$

Finally, we have

$$g(z) = \sum_{n=1}^{\infty} b_n(z-a)^n = (z-a)(b_1 + b_2(z-a) + \ldots + b_n(z-a)^{n-1} + \ldots) = (z-a)h(z),$$

so $h(z) = b_1 + b_2(z_a) + \dots$ and $h(a) = b_1 = g'(a)$. Thus

$$\operatorname{Res}(f/g; a) = \frac{f(a)}{g'(a)}.$$

c) We may write

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n, \ g(z) = \sum_{n=2}^{\infty} b_n (z-a)^n,$$

with $b_2 \neq 0$.

So

$$\frac{f(z)}{g(z)} = \frac{1}{(z-a)^2} \frac{a_0 + a_1(z-a) + a_2(z-a)^2 + \dots}{b_2 + b_3(z-a) + b_4(z-a)^2 + \dots}.$$

Since we want the "-1 coefficient" in the Laurent series expansion of f/g, after factoring out $\frac{1}{(z-a)^2}$ we want the coefficient c_1 of

$$\frac{a_0 + a_1(z-a) + a_2(z-a)^2 + \dots}{b_2 + b_3(z-a) + b_4(z-a)^2 + \dots} = c_0 + c_1(z-a) + c_2(z-a)^2 + \dots.$$

(Notice that this is essentially the method of part a).) Clearing denominators, we get

$$(c_0 + c_1(z-a) + \dots)(b_2 + b_3(z-a) + \dots) = a_0 + a_1(z-a) + \dots.$$

Multiplying out gives

$$a_0 = c_0 b_2,$$

so

$$c_0 = \frac{a_0}{b_2},$$

$$a_1 = c_0 b_3 + c_1 b_2,$$

so

$$c_1 = \frac{a_1 - c_0 b_3}{b_2} = \frac{a_1}{b_2} - \frac{a_0 b_3}{b_2^2}$$

$$= \frac{f'(a)}{g''(a)/2} - \frac{f(a)g'''(a)/6}{(g''(a)/2)^2} = 2\frac{f'(a)}{g''(a)^2} - \frac{2}{3}\frac{f(a)g'''(a)}{g''(a)^2}. \qquad \square$$

Example 8.4. *Let $G(z) = \pi \cot(\pi z)$. This function is analytic on all of $\mathbb{C}$ except for isolated singularities at the zeros of $\sin(\pi z)$. Since the zeros of $\sin z$ are all real and occur at $n\pi$ for $n \in \mathbb{Z}$, the zeros of $\sin(\pi z)$ are all real and occur at n for $n \in \mathbb{Z}$. Since the derivative of $\sin(\pi z)$ is $\pi\cos(\pi z)$ which has zeros at $1/2 + n$ for $n \in \mathbb{Z}$, all the zeros of the denominator are simple and thus all the poles of G are simple. Applying Theorem 8.3 with $f = \pi\cos(\pi z)$ and $g = \sin(\pi z)$, we find that for $n \in \mathbb{Z}$,*

$$\operatorname{Res}(\pi\cot(\pi z); n) = \frac{f(n)}{g'(n)} = \frac{\pi\cos(\pi n)}{\pi\cos(\pi n)} = 1.$$

This example will be of striking use to us later on.

8.3. Applications to integrals.

Example 8.5. *Earlier we computed*

$$I = \int_{-\infty}^{\infty} \frac{dx}{x^4+1},$$

albeit using rather tedious partial fractions computations. We will now redo this example more simply using the Residue Theorem. Much of the strategy is the same: for $R > 1$, let $\gamma_R = \gamma_{1,R} + \gamma_{2,R}$, where $\gamma_{1,R}$ is the straight line path from $-R$ to R and $\gamma_{2,R}$ is the upper semicircular arc from R to $-R$. The singularities of $f(z) = \frac{dz}{z^4+1}$ are at the 4th roots of -1, namely $e^{\pi i/4}$, $e^{3\pi i/4}$, $e^{5\pi i/4}$, $e^{7\pi i/4}$. Of these, precisely the first two lie in the interior of γ_R, so by the Residue Theorem we have

$$\int_R f(z)dz = 2\pi i \left(\mathrm{Res}(f; e^{\pi i/4}) + \mathrm{Res}(f; e^{3\pi i/4}) \right).$$

Since the zeros of z^4+1 are all simple, by Theorem 8.3 we get that

$$\mathrm{Res}(f; e^{\pi i/4}) = \frac{1}{4e^{3\pi i/4}} = \frac{1}{4}e^{5\pi i/4} = \frac{1}{4}e^{\pi i}e^{\pi i/4} = \frac{-1}{4}e^{\pi i/4}$$

and similarly

$$\mathrm{Res}(f; e^{3\pi i/4}) = \frac{-1}{4}e^{3\pi i/4}.$$

So

$$\int_{\gamma_R} f(z)dz = \frac{2\pi i}{-4}(e^{\pi i/4} + e^{3\pi i/4}) = \frac{-\pi i}{2}e^{\pi i/4}(1+i) = \frac{-\pi i}{2}\frac{1+i}{\sqrt{2}}(1+i)$$

$$= \frac{-\pi i}{2\sqrt{2}}(2i) = \frac{\pi}{\sqrt{2}}.$$

The argument that $I = \lim_{R\to\infty} \int_{\gamma_R} f(z)dz$ is the same as before, so we recall it briefly:

$$I = \lim_{R\to\infty} \int_{\gamma_{1,R}} f(z)dz$$

while

$$\lim_{R\to\infty} \int_{\gamma_{2,R}} f(z)dz = 0.$$

Thus

$$I = \frac{\pi}{\sqrt{2}}.$$

Example 8.6. *Let $n \in \mathbb{Z}^+$. We will compute*

$$I = \int_{-\infty}^{\infty} \frac{dx}{x^{2n}+1}.$$

Let us put $J = \int_0^\infty \frac{dx}{x^{2n}+1}$; then evidently

$$I = 2J.$$

The integral can computed using the method of the above example, but because the function $f(z) = \frac{1}{z^{2n}+1}$ has n singularities in the upper half plane, we have to sum over all these residues. So we present the following alternate approach, which is interesting in a somewhat sneaky way. It is taken from [MH, p. 318]. *The motivating idea is to indeed take only $\frac{1}{n}$th of the upper semicircle. So let $R > 1$, and consider the contour γ_R that consists of three pieces: $\gamma_{1,R}$ is the straight line segment from 0 to R, $\gamma_{2,R}$ travels counterclockwise along the circle of radius R centered at 0 from R to $e^{\frac{2\pi i}{2n}} = e^{\pi i/n}$, and $\gamma_{3,r}$ is the straight line segment from $e^{\pi i/n}$ to 0. For future use, let $-\gamma_{3,r}$*

denote this line segment with the reverse orientation, i.e., from 0 *to* $e^{\pi i/n}$. *Now the singularities of* f *lie at* $e^{\frac{2\pi i}{4n}+\frac{k(2\pi i)}{2n}}$ *for* $0 \leq k < 2n$. *Exactly one lies inside* γ_R, *namely* $e^{\pi i/2n}$. *We have*

$$\operatorname{Res}(f; e^{\pi i/2n}) = \frac{1}{\frac{d}{dz}z^{2n}+1}\Big|_{z=e^{\pi i/2n}} = \frac{1}{2ne^{\pi i(2n-1)}2n}$$
$$= \frac{1}{2n}e^{-\pi i}e^{\pi i/2n} = \frac{-1}{2n}e^{\pi/2n}.$$

So by the Residue Theorem we have

$$2\pi i\left(\frac{-1}{2n}e^{\pi i/2n}\right) = \frac{-\pi i}{N}e^{\pi i/2n} = \int_{\gamma_R} f = \int_{\gamma_{1,R}} f + \int_{\gamma_{2,R}} f + \int_{\gamma_{3,R}} f.$$

Now, on the one hand, certainly

$$\lim_{R\to\infty}\int_{\gamma_{1,R}} f = J.$$

On the other hand, the ML-Inequality gives

$$\int_{\gamma_{2,R}} f \leq ML \leq \frac{1}{R^{2n}}\left(\frac{\pi}{n}R\right) \to 0$$

as $R \to \infty$. *We are left with the "third hand",* $\int_{\gamma_{3,R}} f$, *which we write as* $-\int_{-\gamma_{3,R}} f$. *Now* $-\gamma_{3,R}(t) = te^{\pi i/n}$ *for* $t \in [0,R]$, *so* $-\gamma_{3,R}'(t) = e^{\pi i/n}$, *and thus*

$$\int_{\gamma_{3,R}} f = -\int_{-\gamma_{3,R}} f = -\int_0^R \frac{e^{\pi i/n}dt}{(te^{2\pi i/n})^{2n}+1} = -e^{\pi i/n}J.$$

Thus, taking $R \to \infty$ *we get*

$$(-\pi i/n)e^{\pi i/2n} = \lim_{R\to\infty}\int_{\gamma_R} f = (1-e^{\pi i/n})J,$$

so

$$I = 2J = \frac{-2\pi i}{n}\frac{e^{\pi i/2n}}{1-e^{\pi i/n}} = \frac{\pi}{n}\frac{2i}{e^{\pi i/2n}-e^{-\pi i/2n}} = \frac{\pi}{n}\csc(\pi/2n).$$

Example 8.7. *(Fresnel Integrals) Following* [MH, p. 319] *we will show that*

$$\int_0^\infty \cos(x^2)dx = \int_0^\infty \sin(x^2)dx = \sqrt{\frac{\pi}{8}}.$$

In fact, our first order of business is to show that these improper integrals are convergent! This is really not trivial: notice that the functions $\cos(x^2)$ *and* $\sin(x^2)$ do not *tend to* 0 *as* $x \to \pm\infty$. *In the discrete world – i.e., for an infinite series – this would already doom us to divergence. But improper integrals can work differently. The* idea *of the convergence is as follows: the graph of either function is sinusoidal, with amplitude* 1 *but with the peaks and troughs occurring with increasing rapidity (formally, the distance between consecutive zeros approaches* 0 *as* $|x| \to \infty$*). If one contemplates a picture, one guesses that the signed area of each region between consecutive zeros is larger in absolute value than the next one, and the absolute value of the signed area of each piece approaches* 0. *Therefore applying the Alternating Series test to the sum of the signed areas of the "humps" one gets convergence....or so it looks.*

The above argument is actually correct, but there is a variant that is easier to make rigorous: we make the substitution $u = x^2$, *so* $x = \sqrt{u}$ *and* $dx = \frac{1}{2}u^{-1/2}du$. *Then we get*

$$\int_0^\infty \cos(x^2)dx = \frac{1}{2}\int_0^\infty \frac{\cos u du}{\sqrt{u}}$$

(and similarly for the sine integral). Now we still have a sequence of signed areas of "humps" alternating in sign, but each of the humps has the same width, π, and the amplitude is now decreasing to 0. More precisely, since the numerator is periodic and the denominator is increasing, it is now clear that the Alternating Series Test applies to show the convergence of the improper integral.

Having established convergence, we observe that it is equivalent to show

$$\int_{-\infty}^{\infty} \cos(x^2)dx = \int_{-\infty}^{\infty} \sin(x^2)dx = \sqrt{\pi/2}.$$

We do so via the following unlikely looking contour iuntegral. The function we use is

$$f(z) := \frac{e^{iz^2}}{\sin(\sqrt{\pi}z)},$$

and the contour γ_R is a positively oriented rectangle centered at 0 and with sides parallel to the coordinate axes. The width of the rectangle is $\sqrt{\pi}$ and the height is $2R$; thus e.g. the points $\pm\sqrt{\pi}/2$ and $\pm Ri$ are the intersection points of the rectangle with the x and y axes. The singularities of f lie at integer multiples of $\sqrt{\pi}$, and the only such point lying inside γ_R is at 0. This is a simple pole and so we compute

$$\operatorname{Res}(f;0) = \frac{1}{\sqrt{\pi}}$$

and thus

$$\int_{\gamma_R} f = \frac{2\pi i}{\sqrt{\pi}} = 2\sqrt{\pi}i.$$

Now we we will evaluate $\int_{\gamma_R} f$ more directly. Let $\gamma_{1,R}$, $\gamma_{2,R}$, $\gamma_{3,R}$ and $\gamma_{4,R}$ be the four parts of the integral: the first piece proceeds eastward along the bottom side, the second piece northward along the right side, the third peice westward along the top side, and the fourth piece downward along the left side. Our first claim is that

$$\lim_{R\to\infty} \int_{\gamma_{1,R}} f = \lim_{R\to\infty} \int_{\gamma_{3,R}} f = 0.$$

We will do the argument for $\gamma_{1,R}$ as the one for $\gamma_{3,R}$ is almost identical. Along I, we have $z = x - Ri$, so

$$|e^{iz^2}| = |e^{i(x^2-2Rix-R^2)}| = e^{2Rx}$$

and

$$\sin(\sqrt{\pi}z)| = \frac{1}{2}|e^{i\sqrt{\pi}x - R\sqrt{\pi}} - e^{-i\sqrt{\pi}x+R\sqrt{\pi}}|$$

$$\geq \frac{1}{2}\left(|e^{-i\sqrt{\pi}x}||e^{R\sqrt{\pi}}| - |e^{i\sqrt{\pi}x}||e^{-R\sqrt{\pi}}|\right) = \frac{1}{2}\left(e^{R\sqrt{\pi}} - e^{-R\sqrt{\pi}}\right),$$

so

$$|\int_{1,R} f| \leq \frac{2}{e^{R\sqrt{\pi}} - e^{-R\sqrt{\pi}}} \int_{-\sqrt{\pi}/2}^{\sqrt{\pi}/2} e^{2Rx}dx = \frac{1}{R} \to 0$$

as $R \to \infty$. On the other hand, we have

$$\int_{2,R} f + \int_{4,R} f = \int_{-R}^{R} \frac{e^{i(\sqrt{\pi}/2+iy)^2} i dy}{\sin(\pi/2 + \sqrt{\pi}yi)} + \int_{R}^{-R} \frac{e^{i(-\sqrt{\pi}/2+iy)^2} i dy}{\sin(-\pi/2 + \sqrt{\pi}yi)}$$

$$= i\int_{-R}^{R} \frac{e^{i(\pi/4-y^2)}(e^{-\sqrt{\pi}y} + e^{\sqrt{\pi}y})dy}{\cos(i\sqrt{\pi}y)} = 2i\int_{-R}^{R} e^{i(\pi/4-y^2)}dy$$

$$= 2e^{3\pi i/4}\int_{-R}^{R} e^{-iy^2}dy = \sqrt{2}(-1+i)\left(\int_{-R}^{R}\cos(x^2)dx - i\int_{-R}^{R}\sin(x^2)dx\right).$$

Therefore

$$2\sqrt{\pi}i = \sqrt{2}(-1+i)\left(\int_{-\infty}^{\infty}\cos(x^2)dx - i\int_{-\infty}^{\infty}\sin(x^2)dx\right),$$

and thus

$$\sqrt{2}\sqrt{\pi}i = -\int_{-\infty}^{\infty}\cos(x^2)dx + \int_{-\infty}^{\infty}\sin(x^2)dx + i\int_{-\infty}^{\infty}\cos(x^2)dx + i\int_{-\infty}^{\infty}\sin(x^2)dx.$$

Equating real parts gives

$$\int_{-\infty}^{\infty}\cos(x^2)dx = \int_{-\infty}^{\infty}\sin(x^2)dx.$$

In view of this, equating imaginary parts gives

$$\sqrt{2}\sqrt{\pi} = 2\int_{-\infty}^{\infty}\cos(x^2)dx$$

and thus

$$\int_{-\infty}^{\infty}\cos(x^2)dx = \int_{-\infty}^{\infty}\sin(x^2)dx = \sqrt{\frac{\pi}{2}}.$$

8.4. **Summation of series.** In this section we will apply the Residue Theorem to exactly compute the sums of certain infinite series. Especially, we'll show:

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

Theorem 8.8. *(Summation Theorem) Let f be holomorphic on $\mathbb{C}$ except for finitely many isolated singularities. Let C_N be the square centered at 0 with side lengths $2N+1$. Suppose that*

$$\lim_{N\to\infty}\int_{C_N}\pi\cot(\pi z)f(z) \to 0. \tag{21}$$

Let

$$f^*(n) := \begin{cases} f(n) & \text{if } n \text{ is not a singularity of } f \\ 0 & \text{if } n \text{ is a singularity of } f \end{cases}.$$

Then

$$\lim_{N\to\infty}\sum_{n=-N}^{N} f^*(n) = -\sum_{z_i}\operatorname{Res}(\pi\cot(\pi z)f(z); z_i). \tag{22}$$

where z_i ranges over the singularities of f.

Proof. We may assume that N is large enough so that all the singularities of f lie inside C_N. We also observe that $\mathbb{Z}\cap C_N = \varnothing$ and the integers that lie inside C_N are $-N, -N+1, \dots, 0, 1, \dots, N$.

By the Residue Theorem, $\int_{C_N}\pi\cot(\pi z)f(z)dz$ is equal to $2\pi i$ times the sum of the residues of $\pi\cot(\pi z)f(z)$ inside C_N. So

$$0 = \lim_{N\to\infty}\int_{C_n}\pi\cot(\pi z)f(z) = 2\pi i(\Sigma_1+\Sigma_2),$$

where Σ_1 is the sum over the residues of f at $-N, -N+1, \dots, N-1, N$ and Σ_2 is the sum of the residues at the singularities of f. It follows of course that $\Sigma_1 = -\Sigma_2$. We observe that the right hand side of (22) is precisely $-\Sigma_2$. As for the left hand side, if f has a singularity at $n \in [-N, N]$ then $f^*(n) = 0$; if not, then f is holomorphic at n and thus

$$\operatorname{Res}(\pi\cot(\pi z)f(z); n) = f(n)\operatorname{Res}(\pi\cot(\pi z)) = f(n)\cdot 1 = f(n).$$

It follows that the left hand side of (22) is Σ_1, completing the proof. □

Lemma 8.9. *There is a constant $A > 0$ such that: for $N \in \mathbb{Z}^+$, let C_N be the square as in Theorem 8.8. Then for all $N \in \mathbb{Z}^+$ we have*

$$|\cot(\pi z)| \leq A.$$

Proof. Let $z = x + iy$. We consider cases.
Case 1: Supppose $y > \frac{1}{2}$. Then

$$|\cot(\pi z)| = |\frac{e^{\pi iz} + e^{-\pi iz}}{e^{\pi iz} - e^{-\pi iz}}| = |\frac{e^{\pi ix - \pi y} + e^{-\pi ix + \pi y}}{e^{\pi ix - \pi y} - e^{-\pi ix + \pi y}}| \leq \frac{|e^{\pi ix - \pi y}| + |e^{-\pi ix + \pi y}|}{|e^{-\pi ix + \pi y}| - |e^{\pi ix - \pi y}|}$$

$$= \frac{e^{-\pi y} + e^{\pi y}}{e^{\pi y} - e^{-\pi y}} = \frac{1 + e^{-2\pi y}}{1 - e^{-2\pi y}} \leq 1 + e^{-\pi} = A_1,$$

say.
Case 2: Suppose $y < \frac{-1}{2}$. Then

$$|\cot(\pi z)| \leq \frac{e^{\pi y} + e^{\pi y}}{e^{-\pi y} - e^{\pi y}} = \frac{1 + e^{2\pi y}}{1 - e^{2\pi y}} \leq 1 + e^{-\pi} = A_1.$$

Case 3: Suppose $y \in [-1/2, 1/2]$, so $z = N + \frac{1}{2} + iy$ or $z = -N - \frac{1}{2} + iy$. In the first case we have

$$|\cot(\pi z)| = |\cot(\pi(N + \frac{1}{2} + iy))| = |\cot(\pi/2 + \pi iy)| = |\tanh(\pi y)| \leq \tanh(\pi/2) = A_2,$$

say. Similarly, if $z = N + \frac{1}{2} + iy$, we have

$$|\cot(\pi z)| = |\cot(\pi(-N - 1/2 + iy))| = |\tanh(\pi y)| \leq \tanh(\pi/2) = A_2.$$

Thus for all $z \in C_N$, we have $|\cot(\pi z)| \leq \max(A_1, A_2)$. □

Proposition 8.10. *Suppose f is holomorphic on $\mathbb{C}$ except for isolated singularities. If there are $R, M > 0$ such that*

$$|z| \geq R \implies |zf(z)| \leq M,$$

then f satisfies the hypothesis (21) of the Summation Theorem.

Proof. Since $|zf(z)|$ is bounded for all $|z| \geq R$. all the singularities of f lie in $|z| \leq R$. Changing variables $z \mapsto \frac{1}{z}$, we get that $|f(1/z)/z| \leq M$ on $|z| \leq \frac{1}{R}$, so 0 is a removable singularity of $f(z)/z$, so we can write

$$f(1/z)/z = \sum_{n=0}^{\infty} a_n z^n$$

and thus for all $z \geq R$ we have

$$f(z) = a_0/z + a_1/z^2 + \ldots.$$

Next we observe that

$$\int_{C_n} \frac{\pi \cot(\pi z) dz}{z} = 0,$$

since the integrand is an even function with only singularity at 0, so the residue at 0 is 0. Because of this we may write

$$\int_{C_N} \pi \cot(\pi z) f(z) dz = \int_{C_N} \pi \cot(\pi z) \left(f(z) - \frac{a_0}{z} \right) dz.$$

Now for $|z| > R$ we have

$$f(z) - \frac{a_0}{z} = \frac{a_1}{z^2} + \frac{a_2}{z^3} + \ldots$$

Fix $R' > R$. Then, since $a_1 + a_2 z + a_3 z^2 + \ldots$ is analytic on the open disk $|z| < \frac{1}{R}$, it is continuous there and thus continuous on the smaller closed disk $|z| \leq \frac{1}{R'}$ and thus bounded there. It follows that there is $M' > 0$ such that for $|z| \geq R'$ we have

$$|f(z) - \frac{a_0}{z}| \leq \frac{M'}{|z|^2}.$$

Suppose that N is large enough so that all points on the square C_N satisfy $|z| \geq R'$. Then using the ML-inequality and Lemma 8.9 we get

$$\left| \int_{C_N} \pi \cot(\pi z) f(z) dz \right| = \left| \int_{C_n} \pi \cot(\pi z) \left(f(z) - \frac{a_0}{z} \right) dz \right|$$

$$\leq \pi \cdot 4(2N+1) \cdot \frac{M'}{(N+1/2)^2} \sup_{z \in C_N} |\cot(\pi z)| \leq \frac{4\pi M' A(2N+1)}{(N+1/2)^2} \to 0$$

as $N \to \infty$.

□

Exercise 8.1. *Show that the Laurent series expansion of $\pi \cot(\pi z)$ is*

$$\frac{1}{z} - \frac{\pi^2}{3} z - \frac{\pi^4}{45} z^3 - \frac{2\pi^6}{945} z^5 - \frac{\pi^8}{4725} z^7 - \frac{2\pi^{10} z^9}{93555} - \frac{1382\pi^{12}}{638512875} z^{11} + O(z^{13}). \tag{23}$$

Theorem 8.11. *We have*

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Proof. We apply the Summation Theorem with $f(z) = \frac{1}{z^2}$. Note that the hypotheses apply by Proposition 8.10. (Also note that it is much easier to prove Proposition 8.10 for $f(z) = \frac{1}{z^2}$ then to prove the general case.) The only singularity of f is at $f = 0$, so

$$\sum_{n=-N}^{N} f^*(n) = \sum_{n=-N}^{-1} \frac{1}{n^2} + \sum_{n=1}^{N} \frac{1}{n^2} = 2 \sum_{n=1}^{N} \frac{1}{n^2},$$

because $f(z) = \frac{1}{z^2}$ is an even function. Thus

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \lim_{N \to \infty} \sum_{n=1}^{N} \frac{1}{n^2} = \frac{1}{2} \sum_{n=-N}^{N} f^*(n) \overset{\text{ST}}{=} \frac{-1}{2} \operatorname{Res}(\pi \cot(\pi z)/z^2; 0).$$

No problem: the a_{-1} coefficient of $\pi \cot(\pi z)/z^2$ is the a_1 coefficient of $\pi \cot(\pi z)$, so

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{-1}{2} \operatorname{Res}(\pi \cot(\pi z)/z^2) = \frac{-1}{2} \frac{-\pi^2}{3} = \frac{\pi^2}{6}.$$

□

This method generalizes widely:

Theorem 8.12. *(Summing $\zeta(2k)$ for a positive integer k)*
a) Let

$$\pi \cot(\pi z) = \sum_{n=-1}^{\infty} a_n z^n,$$

i.e., a_n is the nth Laurent series coefficient of $\pi \cot(\pi z)$. Then for all $k \in \mathbb{Z}^+$, we have

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{-a_{2k-1}}{2}.$$

b) In particular, we have

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}.$$

$$\sum_{n=1}^{\infty} \frac{1}{n^8} = \frac{\pi^8}{9450}.$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{10}} = \frac{\pi^{10}}{93555}.$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{12}} = \frac{691\pi^{12}}{638512875}.$$

Proof. a) Take $f(z) = \frac{1}{z^{2k}}$ and proceed exactly as in the proof of Theorem 8.11.
b) This follows from part a) and (23). □

Exercise 8.2. *Show that for every even $k \geq 2$, $\sum_{n=1}^{\infty} \frac{1}{n^k}$ is a positive rational number times π^{2k}.*

Notice that since $\pi \cot(\pi z)$ is an odd function, all the even numbered terms in its Laurent series expansion are zero. It follows from the same methods that

$$\sum_{n=-\infty}^{-1} \frac{1}{n^3} + \sum_{n=1}^{\infty} \frac{1}{n^3} = 0$$

and similarly for all odd integers $k \geq 5$. However this is a trivial conclusion – since $\frac{1}{n^3}$ is odd, certainly $\frac{1}{(-n)^3} = -\frac{1}{n^3}$. This method is useless in evaluating the sums

$$\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k}$$

for an odd positive integer k. In fact no exact formula is known for any odd $k \geq 3$. What is known:

Theorem 8.13. *(Odd Zeta Values)*
a) (Apéry [Ap79]*) $\zeta(3)$ is irrational.*
b) (Ball-Rivoal [BR01]*) $\zeta(k)$ is irrational for infinitely many odd k. In fact, the $\mathbb{Q}$-vector space spanned by $\zeta(3), \zeta(5), \zeta(7), \ldots$ is infinite-dimensional.*
c) (Zudilin [Zu04]*) At least one of $\zeta(5)$, $\zeta(7)$, $\zeta(9)$ and $\zeta(11)$ is irrational.*

9. The Riemann zeta function

Unfortunately this section is currently blank.

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