On Some Elementary Invariants of Fields

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Chapter 1

Field Invariants

In this chapter we wish to review a number of classical properties and invariants of fields, and to discuss their elementary (or non-elementary) nature. Actually, we wish to distinguish between two notions of elementary properties of fields. The first, and weaker notion, is that of a property P of fields (in the usual, wide sense of non-formalized mathematics) such that whenever a field F has the property P, so does any other field F' which is elementarily equivalent to F. The second, and stronger notion, is that of a proprerty P of fields which can be given by a sentence ϕ in the language of rings. We call the first property simply *elementary*, or an *elementary invariant*, and the second property *finitely axiomatizable*. By way of explanation, observe that if P is an elementary property, then it can be axiomatized – it is equivalent to a possibly infinite union of first-order sentences ϕ – indeed the definition of an elementary property ensures that the collection of sentences true in every field having property P is well-defined, and this is the axiomatization. If the axioms are themselves equivalent to a finite list, they are equivalent to a single sentence ϕ – this justifies our terminology.

It is very much in the spririt of the talks presented at the Winter School to distinguish between elementary properties and finitely axiomatizable properties. Indeed, a property is elementary if and only if it is preserved by passage to ultra*powers*, whereas a property is finitely axiomatizable if and only if it is preserved by passage to ultra*products*. We will see many examples (some very familiar) of elementary properties that are not finitely axiomatizable below.

A word about the "geometric interpretation of elementary equivalence": as noted by Pop, the condition that two fields F_1, F_2 are elementarily equivalent can be viewed as a sort correspondence between Diophantine problems over one field and Diophantine problems over the other. But we should be careful: this is not literally true in the sense Pop presented in his lecture notes.[1] By definition,

[1]Some discussion of this point took place at Professor McCallum's dining room table.

two fields are elementarily equivalent if for all sentences ϕ (i.e., no free variables) in the language of fields, ϕ holds in one field if and only if it holds in the other. The point is that, since the sentences ϕ will have quantifiers (and no free variables!), there is no corresponding geometric object S_ϕ existing as a subset of $\mathbb{Z}^n$. Moreover, it is certainly *not* enough to check that for every constructible[2] subset $S \subset \mathbb{Z}^n$, $S(F_1)$ is nonempty if and only if $S(F_2)$ is nonempty; we need quantifiers. For example, take $F_1 = \mathbb{C}$ and $F_2 = \mathbb{C}(t)$. Then any finite-type $\mathbb{Z}$-scheme will have points over F_1 if and only if its fibre over $\mathrm{Spec}(\mathbb{Q})$ – a $\mathbb{Q}$-algebraic set – is nonempty, and this is exactly the same condition for it to have points over F_2. On the other hand, the sentence "For all $b, c \in F$, there exists $x \in F$ such that $x^2 + bx + c = 0$" is true in F_1 but not in F_2 (take $b = 0,\ c = t$) – note the use of quantification.

Remark: It is still true that every finite-type $\mathbb{Z}$-scheme S gives rise to a "Boolean-valued" elementary invariant of F, according to whether $S(F)$ is empty or not. We refer to these invariants as *absolute invariants.* As above, when F contains an algebraically closed field, we get no information from the absolute invariants, but otherwise (e.g. when F is finitely generated) we do get some information; some particular examples of this will be discussed later, including an example of two finitely generated fields which are *not* distinguished by any of their absolute invariants.

1.1 Finitude, characteristic, algebraic closure

Invariant 0: Finiteness/Infinitude of F.

Because of the sentence "There exist $1 + \ldots + 1$ (q times) distinct elements of F and there do not exist $1 + \ldots + 1$ ($q + 1$ times) distinct elements," having q elements is a finitely axiomatizable property of a field. It follows that finiteness is an elementary property, being the infinite union of these sentences over all q. On the other hand, finiteness itself is not finitely axiomatizable, as we can see by considering the pseudofinite field associated to a nonprincipal ultrafilter D on the prime numbers.

$$\mathbb{F}_\infty(D) := \prod \mathbb{F}_p/D.$$

(If finiteness were finitely axiomatizable, then $\mathbb{F}_\infty(D)$ would be finite, but then it would be finite of some given cardinality p, which it isn't, because at most one of its factors has this property.)

Of course, the above argument is quite general: if ψ_i is an infinite collection of

[2]In the Zariski topology; recall that a constructible subset of a topological space is a finite union of locally closed subspaces, or equivalently is an element of the Boolean algebra generated by the closed subsets

sentences with the property that for all i there exists a structure M_i satisfying ψ_i and the negation of ψ_j for all $j \neq i$, then $\bigvee_i \psi_i$ is not finitely axiomatizable by consideration of the ultraproduct $\Pi M_i/D$.[3]

Let us make one more comment before moving on to more interesting examples. First of all, our Invariant 0 has classified finite fields up to isomorphism as well as elementary equivalence – the complete theory of a finite field has a unique model. It is worth remarking that the analogous statement is false for any infinite field, and not just for cardinality reasons:

Proposition 1 *The complete theory of any infinite field has at least two countable models.*

Reminder of proof: One knows that in a complete theory with a unique countable model (an "ω-categorical theory"), the algebraic closure of any finite set is finite, and indeed uniformly bounded: there exists a function $f : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ such that the algebraic closure of an n-element set has at most $f(n)$ elements [Marker, p. ???]. Applying this to an ω-categorical field, we see that there exists $N = N(F)$ such that for all $\alpha \in F$, the subfield generated by α has size at most N. But this means that $F \subset \mu_N(F)$, so that F has at most N elements.

Using a little more model theory, this result can be improved as follows.

Proposition 2 *Any infinite field has infinitely many countable models.*

A proof of this result was supplied at the author's request by N. Ackerman; we give his argument in the appendix.

Remark: The number of countable models of any theory over a countable language is at most $2^{\aleph_0}$, so that the number of countable models $I(T(F), \aleph_0)$ of any infinite field F satisfies

$$\aleph_0 \leq I(T(F), \aleph_0) \leq 2^{\aleph_0}.$$

These inequalities are sharp: e.g. the lower bound is attained by an algebraically closed field and the upper bound by a real-closed field. (We are not sufficiently set-theoretically inclined to pursue here the question of whether, assuming the falsity of the continuum hypothesis, $I(T(F), \aleph_0)$ can take on a cardinality strictly between $\aleph_0$ and $2^{\aleph_0}$ except to say: probably not.)

Note that the elementary equivalence class of the pseudofinite field $\mathbb{F}_\infty(D)$ depends on the choice of the ultrafilter D! (We see again the basic difference between ultraproducts and ultrapowers – it follows immediately from Los' theorem that all ultrapowers of a structure are elementarily equivalent to the structure.) For instance, consider the sentence

[3] This formulation makes especially clear that one use of ultraproducts is to hide appeals to the compactness theorem.

"There exists $x \in F$ such that $x^2 + 1 = 0$."

This is true in the field $\mathbb{F}_p$ iff $p = 2$ or p is one modulo 4 – i.e., on an infinite, coinfinite subset of $\mathcal{P}$, the set of prime numbers. Indeed, inside the Stone-Cech compactification of $\mathcal{P}$ both the collection of D's for which this statement is true in $\mathbb{F}_\infty(D)$ and the the set of D's for which it's false, are nonempty open subsets.

Exercise: As D varies through the $2^{2^{\aleph_0}}$ elements of $SC(\mathcal{P})$, how many elementarily inequivalent fields $\mathbb{F}_\infty(D)$ arise?

Invariant 1: The characteristic.

The discussion is exactly as above: having a given positive characteristic p is finitely axiomatizable, so having positive characteristic and having characteristic zero are both elementary, but neither of these are finitely axiomatizable, as the example $\mathbb{F}_\infty(D)$ makes clear.

Remark about absolute invariants: Let $p \geq 0$ be a prime ideal of $\operatorname{Spec}\mathbb{Z}$, and let $S_p = \operatorname{Spec}\mathbb{F}_p \subset \operatorname{Spec}\mathbb{Z}$ be the subscheme given by the ideal p (so $\mathbb{F}_0 = \mathbb{Q}$.) Then $S_p(F)$ is the set of F-algebra homomorphisms from $\mathbb{F}_p \otimes_{\mathbb{Z}} F$ to F; this is the empty set unless F has characteristic p. Thus the characteristic of a field is an absolute invariant, and, knowing the characteristic, we can reduce the study of all absolute invariants to the the study of finite-type schemes over the prime subfield of F.

Invariant 2: Whether F is algebraically closed.

Again, being algebraically closed is elementary but the obvious axiom scheme cannot be made finite. For instance, this can be seen as follows: for each prime p, let F_p be a perfect field whose absolute Galois group is a nontrivial pro-p-group. Then $\prod F_p/D$ is algebraically closed.[4]

Coming back to the question of classification up to elementary equivalence versus classification up to isomorphism, recall that algebraically closed fields are classified up to isomorphism by their characteristic and their absolute transcendence degree. In particular, ACF_p is uncountably categorical, so by Vaught's test it is complete. (Of course, the "better" proof is to deduce the completeness from quantifier elimination!) Thus we have found enough invariants to classify algebraically closed fields up to equivalence, and again the relation between equivalence and isomorphism is quite easy – the extra invariant needed is the transcendence degree.

[4]Although we shall not mention it again in these notes, it is important that the obvious axiom scheme for algebraically closed fields, although infinite, is *recursive*, an important point for the decidability of the theory ACF_p of algebraically closed fields of characteristic p.

1.2 Pseudoalgebraic closure

Invariant 3: Whether F is PAC.

Recall that a field F is said to be pseudo-algebraically closed (PAC) if every absolutely irreducible variety V/F has an F-rational point. This is an elementary invariant: to see this easily, we use the result of Frey-Geyer ([**?**]), which says that it suffices to check that every absolutely irreducible plane curve C/F has an F-rational point. The point is that it is elementary to say whether a plane curve of degree d is irreducible, and if it has degree d then geometrically it can break up into at most d components, so that if it is absolutely reducible, it becomes so after a degree d field extension. We will see later on that finite field extensions can be interpreted in F; the conclusion is that "Every absolutely irreducible degree d plane curve has an F-rational point" is given by a sentence in the language of rings. As usual, the PAC property, being the union over d of all these sentences, is not finitely axiomatizable, and again the pseudofinite field $\mathbb{F}_\infty(D)$ gives the counterexample: no finite field is PAC (it is very easy to construct a hyperelliptic curve over a finite field without rational points), but the Riemann hypothesis for curves over finite fieldsd implies that for fixed d, absolutely irreducible plane curves of degree d have rational points whenever the cardinality of $\mathbb{F}$ is sufficiently large.

I'm certainly no expert on the matter, but my impression is that while much is known about PAC fields ([**?**]), there are too many of them to classify even up to elementary equivalence: for each projective profinite group G, there is a PAC field F with Galois group $G_F = G$. Morever, there will be many different PAC fields with Galois group G – indeed, in the (simplest!) case $G = \hat{Z}$, we already saw that there are infinitely many PAC fields $\mathbb{F}_\infty(D)$ with this Galois group.

1.3 The Hilbert property

Invariant 4: Whether F is Hilbertian.

The property which from the perspective of [**?**] is somehow dual to PAC, namely, Hilbertianity of F, is also elementary. To see this, we use the geometric characterization in terms of thin sets from [**?**]. Namely F is *not* Hilbertian if $F = \mathbb{A}^1(F)$ can be written as a finite union $F = S_1 \cup \ldots \cup S_n$, where $S_i = \phi_i(C_i(F))$, $\phi_i : C_i \to \mathbb{A}^i$ is a morphism of degree at least 2 from an irreducible affine curve.

Some examples of non/Hilbertian fields: A finite field is not Hilbertian; nor is an algebriacally closed field. Any field for which there exists n such that the set of nth power classes F^*/F^{*n} is finite is not Hilbertian – so no locally compact field is Hilbertian. A fundamental property of a Hilbertian field is that any finite group G arising as a Galois group of $F(T)$ also arises as a Galois group of infinitely many disjoint field extensions K_i/F. Since it is conjectured that for *any* field F, every finite group arises as a Galois group of $F(T)$, it is also conjectured that every finite group arises infinitely many times as a Galois group over every Hilbertian field.[5] A number field is Hilbertian (Hilbert's theorem!); for any field F, $F(T)$ and $F((T_1, T_2))$ are Hilbertian; and finitely generated extensions of Hilbertian fields are Hilbertian [**?**]. In particular, all absolutely finitely generated fields of characteristic zero and all infinite finitely generated fields of characteristic p are Hilbertian. We mention in passing a surprising connection between the seemingly antithetical properties of Hilbertianity and pseudo-algebraic closure: with probability 1, the fixed field of the subgroup generated by n randomly chosen elements of the absolute Galois group of a Hilbertian field is PAC [**?**]. In terms of our attempt to classify fields up to elementary equivalence, the Hilbert property gives us only the following (rather weak) conclusion:

Fact: If $F \sim F(T)$, then F is Hilbertian.

1.4 The C_i property

One says that a field K has the property $C_i(d)$ if every homogeneous form of degree d in more than d^i variables has a (not identically zero) solution. This is visibly finitely axiomatizable. So the property C_i, which is by definition $C_i(d)$ for all d, is clearly elementary. This is a key property in the context of elementary equivalence of function fields, so let us review some examples and facts:

Fact: A field is C_0 if and only if it is algebraically closed.

Fact (Chevalley-Warning): A finite field is C_1.

Fact (Lang): A complete local field with algebraically closed residue field is C_1.

Fact: If F is C_i, $F((t))$ is C_{i+1}.

Fact (Tsen-Lang): If F is C_i and K/F has transcendence degree n, then K is C_{i+n}.

Fact (CITE!!): If F is not C_{i-1}, $F(T)$ is not C_i.

This last fact shows that the transcendence degree of a finitely generated function field over an algebraically closed field is an elementary invariant. We will give another proof of this using Brauer groups.

Example: The property C_2 is not finitely axiomatizable, as the theorem of

[5] One knows at least that every finite solvable group arises over $F(T)$; for similar partial results, see [**?**].

Ax-Kochen shows: there is a nonprincipal ultrafilter D on $\mathcal{P}$ such that

$$\prod \mathbb{Q}_p/D \cong \prod \mathbb{F}_p((t))/D.$$

For all d and p each field $\mathbb{F}_p((t))$ is $C_2(d)$, so the right hand side is $C_2(d)$ for all d, hence C_2. Terjanian showed that no $\mathbb{Q}_p$ is C_2, so if C_2 were given by a sentence, the left hand side would not be C_2, contradiction.

Exercise: Show that for all $i > 0$, C_i is not finitely axiomatizable.

1.5 Finite-dimensional F-algebras

We now wish to explore a constellation of elementary invariants arising from interpreting finite-dimensional algebras in F. Recall that we say an algebraic structure is *interpretable* in F if we can realize the set as a definable (with constants!) subset of F^n for some n in such a way that the functions, relations, etc. on the structure are also definable with respect to F.

In particular, any finite dimensional F-algebra A can be interpreted in F; if it has dimension n we need n^3 "structure constants" for A to define the multiplication map: let $A = Fv_1 \oplus \dots Fv_n$ and put

$$(\Sigma a_i v_i) \cdot (\Sigma b_j v_j) := \Sigma_k \Sigma_i \Sigma_j a_i b_j c_k^{ij} v_k.$$

The following properties of a finite-dimensional algebra A are finitely axiomatizable:
a) A has a unit.
b) A is associative.
c) A is commutative.
d) The center of A is $Fv_1 = F1$.
e) A is a division algebra.
f) A is isomorphic to the matrix algebra $M_n(F)$.
g) $\mathrm{Aut}_{F-alg}(A)$ is a (particular!) finite group G.

All of these are clearly elementary statements, except possibly for the last: if we can interpret A we can interpret $\mathrm{End}_F(A) = M_{[A:F]}(A)$, the algebra of F-vector space endomorphisms of A. We can then write down the conditions for an element $\phi \in \mathrm{End}_F(A)$ to preserve the algebra structure and to be invertible, so if we have finitely many of them we can explicitly write down that under composition they form a group isomorphic to some given group G.

Absolute algebraic invariants: Since the notion of interpretation allows constants, the class of finite-dimensional algebras that we can speak of individually is that class which arises by base change from the algebraic closure of the prime subfield (because, being finite-dimensional, they must therefore arise by basechange from a finite field extension of the prime subfield). For example, we can speak individually of the finite field extensions of the prime subfield of F, and from this we deduce that the **asbsolute subfield** is an elementary invariant. This is an example of one of our absolute invariants.

We can't speak of other F-algebras individually, but by quantifying over them we can make many statements about the non/existence of finite dimensional F-algebras satisfying certain properties.

Galois groups: For instance, the above work shows that for any finite group G, the statement "There exists a finite Galois extension of F with Galois group G" can be given by a sentence in the language of fields. With just a little more work using properties of tensor products of field extensions (exercise!), one can say "There exist n linearly disjoint Galois extensions $K_1, \ldots, K_n$ over F with Galois group G." Because of this, we can attach to the elementary equivalence class of a field F the data of a *multiplicity function*

$$m_F = m_{G_F} : \mathcal{G} \to \{0, 1, \ldots, \infty\},$$

where $\mathcal{G}$ is the set of isomorphism classes of finite groups – to each finite group G, $m_F(G)$ is the number of Galois extensions of F with Galois group G, or ∞ if there are infinitely many.

A good question to ask at this point is: can we extract further elementary information from G_F besides the multiplicity function m_F? The answer depends upon "the size" of G_F. Recall that a profinite group G is said to be *small* if it only has finitely many subgroups of any given finite index. Having small absolute Galois group is an elementary invariant – indeed it is equivalent to all the multiplicities being finite (and the fact that there are "only" finitely many isomorphism classes of groups of fixed order n). We leave for the reader the proof of the following

Fact: If $m : \mathcal{G} \to \{0, 1, \ldots\}$ (no ∞!) is a small multiplicity function, there exists at most one isomorphism class of profinite group G with $m = m_G$.

So if G_F is small, it is itself an elementary invariant. Disappointingly, the converse is also true: if $m_F(G) = \infty$, then for all cardinals κ, there exists an elementarily equivalent field which has at least κ^+ disjoint extensions with Galois group G. (The proof is straightforward for those who are comfortable with saturated models: realizing G as a transitive subgroup of some symmetric group S_n, by realizing a certain n-type over a cardinality κ subset, one sees that any κ^+-saturated model will do.)

The number of elementary equivalence classes of fields with G_F isomorphic to a fixed G will in general be large – we saw this before with the many inequivalent fiels $\mathbb{F}_\infty(D)$, all with Galois group $\hat{Z}$. (There are many more inequivalent fields with this Galois group and characteristic zero, e.g. the non-PAC field $\mathbb{C}((t))$.) In fact the class of fields which are determined up to elementary equivalence by their absolute Galois group has been determined by F.V. Kuhlmann, who finds, apart from the classical example of real-closed fields and certain p-adically closed fields (among them the abelian extensions $K/\mathbb{Q}_p$) there are only three possible further families of such fields.

If we restrict to finitely generated fields, however, then it is a celebrated result of Pop that $G_F \cong G_{F'} \implies F \cong F'$ – this is the so-called zero-dimensional case of "Grothendieck's anabelian dream." Thus the absolute Galois group of a finitely generate field "has enough information" to determine the field; we just need to extract that information in an elementary way!

Notice that our multiplicity function is useless for any Hilbertian field: if the inverse Galois problem has an affirmative solution, the multiplicity function on any Hilbertian field will be identically ∞-valued!

Brauer group invariants: Recall that the Brauer group of a field classifies finite-dimensional central simple F-algebras up to equivalence: every such algebra A is isomorphic to $M_n(D)$, a matrix algebra over a division algebra, and $A \sim A'$ if $D \sim D'$. Recall also that the Brauer group of a finite field vanishes, whereas the Brauer group of a number field is calculated by the Hasse principle:

$$0 \to Br(K) \to \bigoplus_v Br(K_v) \stackrel{\Sigma}{\to} \mathbb{Q}/\mathbb{Z} \to 0$$

where the middle sum extends over the places of K, and the Brauer group of the completion K_v is $\mathbb{Q}/\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}$ or 0 according to whether v is a finite place, a real Archimedean place, or a complex Archimedean place.

Say that K is a field of characteristic zero. There is a natural (basechange) map

$$\varphi : Br(\mathbb{Q}) \to Br(K)$$

induced by $A \mapsto A \otimes_{\mathbb{Q}} K$. Since finite-dimensional algebras over $\mathbb{Q}$ can be spoken of in absolute terms, the kernel of φ is an elementary invariant; we call it the *absolute Brauer kernel*.

In fact this is an example of what we called above an absolute geometric invariant. Associated to every element of $D \in Br(\mathbb{Q})$ we have a Severi-Brauer variety $V_D/\mathbb{Q}$, with the property that for any field K of characteristic zero, $[D \otimes K] = 0 \in Br(K)$ iff $V_D(K) \neq \emptyset$. Thus the absolute Brauer kernel is obtained by checking whether each Severi-Brauer variety over $\mathbb{Q}$ has a K-rational point.

In fact we can do the same thing with $\mathbb{Q}$ replaced by any number field k. As an application, let K/k be a finitely generated field of characteristic zero with absolute subfield k, so that $K = k(V)$ can be viewed as the function field of a smooth projective absolutely irreducible k-variety V. The Brauer kernel then has an interpretation via the exact sequence

$$0 \to \mathrm{Pic}(V) \to \mathrm{Pic}(V/k)(k) \xrightarrow{\alpha} Br(k) \to Br(V),$$

where the map α gives the obstruction to a rational divisor class being represented by a k-rational divisor. Since the map $Br(k) \to Br(K)$ factors through $Br(V)$, the existence of a nontrivial element of the absolute Brauer kernel is equivalent to the existence of a k-rational divisor class on V not represented by a k-rational divisor. We conclude that this (admittedly somewhat abstruse) geometric property is an elementary invariant of K.

As an application of the absolute Brauer kernel, we can distinguish Severi-Brauer varieties of the same dimension from each other: indeed, it is a fundamental result of Amitsur that if $K = k(V_D)$ is the function field of the Severi-Brauer variety D, the absolute Brauer kernel is the cyclic subgroup generated by $[D]$. It follows that if $K \sim K'$ are the function fields of two Severi-Brauer varieties of the same dimension $\langle [D] \rangle = \langle [D'] \rangle$. Amitsur conjectured that whenever two Brauer group elements generate the same cyclic subfield of the Brauer group, their function fields are isomorphic, and he proved this in case the division algebras in quiestion are cyclic. Since one knows that all division algebras over global fields are cyclic (Hasse-Brauer-Noether theorem), this completes the proof.

1.6 Transitional invariants

In this section we axiomatize (in the informal sense!) a kind of field invariant which "changes properly" under finitely generated field extensions. A model of the sort of behavior we have in mind is the C_i property for a field: namely if k is a C_i field and K/k is a field extension of transcendence degree r, then K is C_{i+r}. This leads us to the following

Definition: A **transitional field invariant** is an assignment

$$i : \text{Fields} \to \{0, 1, 2, \ldots, \infty\}$$

with the following properties (we interpret the relations $\leq$ and $+$ on the set $\{0, 1, 2, \ldots \infty\}$ in the most obvious way):
- If L/K is an algebraic field extension, then $i(L) \leq i(K)$.
- If L/K is a field extension of transcendence degree n, then $i(L) \leq i(K) + n$.

A transitional field invariant is **strict** if whenever K/F is a finitely generated field extension of transcendence degree n such that $i(F) < \infty$, then $i(K) = i(F) + n$.

If i is any transitional field invariant, then we define its **virtualization** vi to be $vi(F) := \min_{[K:F]<\infty} i(K)$. If i is strict, then so is vi.

Analogously, we say that a $\{0, 1, \ldots, \infty\}$-valued field invariant is **local** if whenever K is a complete field with residue field F we have $i(K) \leq i(F) + 1$; it is **equicharacteristic local** if we have $i(F((t))) \leq i(F) + 1$. Finally, it is **strict** if we have equality when $i(F) < \infty$.

The theme of this section is the search for a strict transitional invariant i which is elementary and which is finite on a large class of fields (especially, on finitely generated fields). The existence of this implies that the transcendence degree is an elementary invariant among finitely generated fields. We will see several examples of elementary invariants which are conjecturally strictly transitional and strictly transitional invariants which are conjecturally elementary. In the end it is the Milnor conjecture which provides us with an invariant which has all the desirable properties.

Example 0: The trivial example of a strict transitional invariant is the absolute transcendence degree. But it is not elementary (nor is it local, for that matter).

Example 1: The "Tsen-Lang invariant" $TL : F \mapsto$ the least i such that F is a C_i field is elementary strict transitional. It is moreover equicharacteristic local (but not local!). So, as we've seen above, TL allows us to conclude that transcendence degree is an elementary invariant among function fields over e.g. algebraically closed fields and finite fields. The problem is that being C_i for any i is too strong a property: $vTL(\mathbb{Q}_p) = vTL(\mathbb{Q}) = \infty$.

Example 2: The p-cohomological dimension. Let $cd_p(F)$ be the p-cohomological dimension of G_F: this is the unique i such that for all discrete p-primary torsion G_F-modules M and all $n > 0$, $H^{i+n}(G_F, M) = 0$ and $H^i(G_F, M) \neq 0$ for some M. For simplicity, we stay away from the case when p equals the characteristic of F. Under this hypothesis, one finds in [**?**] that cd_p (and hence also vcd_p) is strict transitional and strict local. Moreover, it has very appealing finiteness properties:

- $cd_p(F) = 0$ iff F is p-closed.
- For all p, $cd_p(\mathbb{F}_q) = 1$.
- For all p, ℓ, $cd_p(\mathbb{Q}_l) = 2$.
- $cd_2(\mathbb{R}) = \infty$, but $vcd_2(\mathbb{R}) = 0$.
- If F is a global field, $cd_p(F) = 2$ unless $p = 2$ and F is formally real, in which case $cd_2(F) = \infty$. But for every global field, $vcd_p(F) = 0$.

This brings us to the following

Question 3 *Is cd_p an **elementary** invariant?*

Observe that if any invariant is elementary, so is its virtualization. So if the answer to this question is yes, then we can detect transcendence degree over an enormous variety of fields. Here are some results in this direction:

Having $cd_p(F) = 0$ is elementary. (Because admitting a degree p field extension can be seen from the "Galois invariant" of the previous section.)

Having $cd_p(F) = 1$ is elementary. Indeed, one knows that having $cd_p(F) = 1$ is equivalent to the vanishing of the p-component of the Brauer group of all finite extensions of F; see [**?**]. We have seen that this latter property is elementary.

Exercise: Show that if F contains p pth roots of unity, the property of $Br(F)[p^\infty] = 0$ is finitely axiomatizable. (Hint: use the Merkurjev-Suslin theorem.) Is the property $cd_p(F) \leq 1$ finitely axiomatizable?

One defines simply $cd(F)$ as the supremum over $cd_p(F)$ for all p. So if cd_p is elementary for all primes p then so is cd, but not necessarily conversely. One knows that the property of having cohomological dimension 2 is elementary: this is a consequence of a deep theorem of Suslin, which says that cohomological dimension at most 2 is equivalent to the property that for all finite extensions l/k and all finite central division algebras D/l, the *reduced norm* map $N : D \to l$ is surjective. We leave it as an exercise to the reader who is familiar with this material to show that this latter statement is elementary.

I know of no similar (even conjectural) characterizations of having p-cohomological dimension at most 3 that would give us a truly good reason for believing its general elementary nature, although I will admit that in my heart I am convinced by the assembled evidence together with the following remarkable fact.

Proposition 4 *Because the Milnor Conjecture holds, cd_2 is an elementary invariant.*

Proof: The particular form of the Milnor Conjecture we want is (as found on p. 12 of Pop's notes) that, for any field F of characteristic different from two, the correspondence

$$e_n : I^n(F)/I^{n+1}(F) \to H^n(F, \mathbb{Z}/2\mathbb{Z})$$

induced at the level of Pfister forms by

$$(1, a_1) \otimes \ldots \otimes (1, a_n) \to \chi_{-a_1} \cup \ldots \cup \chi_{-a_n}$$

is an isomorphism. Here the left hand side is tensor (Kronecker) product of n diagonalized binary quadratic forms, and the χ_{-a_i} in the right hand side denotes the image of $-a_i \in F^\times/F^{\times 2}$ under the Kummer isomorphism $F^\times/F^{\times 2} \to H^1(F, \mu_2) = H^1(F, \mathbb{Z}/2\mathbb{Z})$. Since one knows that the p-cohomological dimension can be computed using the Galois module $\mathbb{Z}/p\mathbb{Z}$ [CG, p. ??], it follows that the 2-cohomological dimension of any field k (not of characteristic 2) is the largest integer n such that there exists an anisotropic n-fold Pfister form. But, the statement "There exists an n-fold anisotropic Pfister form" (for a particular value of n) is evidently a sentence in the language of fields, completing the proof.

1.7 More Brauer invariants

In this section, we work with an unspecified prime p which we assume *not* to be the characteristic of any of the fields under discussion.

The Milnor conjecture implies that the transcendence degree is an elementary invariant among function fields over a field k with $(v)cd_2(k) < \infty$. Here we want to present a conjecture about Brauer groups that implies the same result for any odd prime p.

Further Brauer invariants: Define the **period** of a central division algebra D/K to be the order of its class in $Br(K)$ – more concretely, it is the least positive a such that $D^{\otimes a} \cong M_n(K)$. Also we define the *index* of D/K just to be $\sqrt{[D:K]}$. One knows that for any division algebra D over any field K, the period divides the index and the two quantities have the same prime factors. For any given prime p and positive integer a, the following can easily be made into a sentence in the language of fields:

$\Psi(p, a)$:= "There exists a K-central division algebra of period p and index p^a."

It is probable that these sentences should distinguish between function fields of differing transcendence degrees over a very general class of base fields k, as we will now explain. We begin with the following result:

Proposition 5 *Let k be a field with the property that for some prime p, there exists an absolute bound on i on the indices of all elements of prime order in the Brauer group of all finite extensions of k. Then k is not elementarily equivalent to any finitely generated regular field extension of positive transcendence degree.*

Seeking a contradiction, suppose K/k is a finitely generated field extension of positive transcendence degree with $K \sim k$. Because the hypotheses are stable upon finite base extension of k, we may assume that k contains the pth roots of unity and that K admits a corresponding smooth projective variety V/k with a k-rational point, so that the map $Br(k) \to Br(K)$ is injective. By hypothesis, there is a k-central division algebra D of period p and index p^i but none of period p and index p^{i+1}. We will produce a central division algebra D'/K of period p and index p^{i+1}, exhibiting the inequivalence of k and K. First consider the pushforward of D to K – we get a "constant" Azumaya algebra, which by our above base extension we have ensured is still division. Observe that there exist plenty of cyclic p extensions L/K such that D_L remains a division algebra – indeed take *any* p-extension except a basechange of k. The property of their existing a period p index p^i division algebra and a cyclic p-extension which does not reduce its index is *elementary* – since it's true in K, it must therefore also be true of some division algebra (which we continue to denote D) over k. Now consider the following result from [**?**]:

Proposition 6 *Let k'/k be a cyclic p-extension with generator σ and A/k a central simple algebra. Write $A_{k'}$, $A_{k(t)}$, $A_{k((t))}$ for the base changes of A to k', $k(t)$ and $k((t))$ respectively. Then the following are equivalent:*
a) $A_0 := A_{k'}$ is a division algebra.
b) $A_1 := A_{k(t)} \otimes_{k(t)} (k'(t)/k(t), \sigma, t)$ is a divison algebra.
c) $A_2 := A_{k((t))} \otimes_{k((t))} (k'((t))/k((t)), \sigma, t)$ is a division algebra.

Choose a finite map $V \to \mathbb{P}^n$ such that in the associated coordinate system $(t_1, \ldots, t_n)$ the assumed k-rational point on V lies over the origin in $\mathbb{P}^n$. By an iterated application of the propostion, we can build a division algebra $D_n/k(t_1, \ldots, t_n)$ of period p and index p^{i+n}. Looking at part c) of the proposition, we see that D_n remains a division algebra even locally around the origin, so that its pullback to V remains a division algebra even locally around P , hence is a division algebra.

The question of course, is when the hypotheses of Proposition 5 are satisfied. Consider the following conjecture.

Conjecture 7 *(Period-index conjecture for Brauer groups) Let k be a field of characteristic different from p with $cd_p(k) = d$.*
a) If $[D] \in Br(k)[p]$, then the index of D divides p^{d-1}.
b) Assume that there exists a positive integer a such that for all finite field extensions l/k, all elements $[D] \in Br(l)[p]$ have index dividing p^a. Then if K/k is a field extension of transcendence degree one, all elements $D \in Br(K)[p]$ have index dividing p^{a+1}.

Proposition 5 and Conjecture 7 imply that for any p, fields of differing transcendence degrees over a field of finite (virtual) p-cohomological dimension are elementarily inequivalent.

I am afraid that the evidence for Conjecture 7 is thus far rather meager. Part a) holds rather trivially for fields of p-cohomological dimension at most one. It is classically known to hold for local and global fields; it is known for $p = 2$ and $p = 3$ for C_2-fields [Artin];