HANDOUT SEVEN: CONSERVATIVE VECTOR FIELDS AND A FUNDAMENTAL THEOREM

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1. Introduction: Newton's vector field

The motivation for this unit is to make mathematical sense out of our idea that in a gravitational field energy is conserved. More precisely, consider the vector field $F = -F_2(x,y,z) = -(x,y,z)/||(x,y,z)||^3$, which is, according to Newton's law of universal gravitation and up to a multiplicative constant, the force felt by a mass at a point (x,y,z) due to the gravitational attraction of a large mass located at the origin. Suppose we take any path through space so that the distance to the origin is some constant d: that is, we take any path confined to the sphere of radius d. Then the velocity vector at every point lies in the tangent plane to the sphere S_d, and in particular is perpendicular to the purely radial vector field F: in other words, the line integral $\int_C F \cdot d\mathbf{r}$, where C is any curve confined to the sphere S_d, is equal to zero.

But, as mentioned in the last handout, we suspect that more is true: suppose we take an arbitrary *closed* path C (not passing through the origin), e.g. an elliptical orbit. Then the field F need not be perpendicular to the velocity vector at any point, so the integrand of the line integral $\int_C F \cdot \,\dot{\mathbf{r}}$ is not identically zero, but we nevertheless feel that the work done should come out to be zero. More generally, if we take a path C which is not even necessarily closed (but still not passing through the origin) with initial point $P_0 = (x_0,y_0,z_0)$ and $P_1 = (x_1,y_1,z_1)$, then we feel that the total work should be given by an expression which depends only on the distances $d_0 = ||(x_0,y_0,z_0)||$ and $d_1 = ||(x_1,y_1,z_1)||$ of P_0 and P_1 from the origin: if $d_1 > d_2$, then the particle ends up farther away than it starts, so energy must be put into the system: the work should be negative. Conversely, if $d_2 > d_1$, the particle ends up closer than it started, so energy is released from the system: the work should be positive. And if $d_1 = d_2$, energy should be conserved.

We could compute some line integrals around various closed paths and see that they are zero, but even if we did this for many paths, how would we know that the line integral is zero for *all* closed paths? We need to instead take a more abstract look at properties of vector fields in general: it turns out that there are several conditions on a vector field F which are *equivalent* to the line integral around every closed path being zero, one of which we can verify in a straightforward way. As a consequence of this study, we will discover a wonderful generalization of the fundamental theorem of calculus to integrals along closed paths, and this will at the same time make clear exactly what kind of "energy" is being conserved (potential energy!)

2. Conservative vector fields

Let F be a vector field defined on the plane or in space – in fact, as usual we really only require it to be defined on some appropriate subset thereof: for instance, even Newton's vector field $-\mathbf{r}/||\mathbf{r}||^3$ has a singularity at the origin. Let us give a name to the (possible) property of F that we are trying to understand:

We say F is **conservative** if for *every* closed path C on which the vector field is defined, the line integral $\int_C F \cdot d\mathbf{r} = 0$.

The point is that the condition of being conservative turns out to be equivalent to many other conditions, some relatively obviously so, others not. Here is another condition:

A vector field F is said to be **independent of path** if whenever C_1 and C_2 are two oriented paths with the same starting point P and ending point Q, then

$$\int_{C_1} F \cdot d\mathbf{r} = \int_{C_2} F \cdot d\mathbf{r}.$$

Again, this is a condition that is difficult to verify (how will we test all possible paths), but on the other hand at least we can sometimes show that it does not hold:

Example: Let $F(x, y) = (-y, x)$. Recall that this is one of our vector fields with circular integral curves, and this field in particular has constant curl $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = d(x)/x - d(-y)/dy = 2$. Take $P = (-1, 0)$ and $Q = (1, 0)$. Here are two different paths from P to Q: C_1 is the straight line path, with a parameterization given by $\mathbf{r}_1(t) = (-1 + t, 0)$ for $0 \leq t \leq 2$; C_2 is the positively oriented semicircular arc given by $\mathbf{r}_2(t) = (\cos t, \sin t)$, for $\pi \leq t \leq 2\pi$. In fact we can see before we even do the calculation that the first line integral will be zero: along the right half of the straight line the vector field points due north, along the left half of the straight line the vector field points due south, and there is a symmetry here so that the magnitude at $(-x, 0)$ is equal to the magnitude at $(x, 0)$. We have also seen and computed before that the line integral along C_2 will not be zero: we are moving along an integral curve of the vector field, so the work done will be positive. But to be sure:

$$\int_{C_1} -ydx + xdy = \int_{-1}^{1} 0 + (-1 + t) \cdot 0 = 0.$$

$$\int_{C_2} -ydx + xdy = \int_{\pi}^{2\pi} -\sin t(-\sin t dt) + (\cos t)(\cos t dt) = \int_{\pi}^{2\pi} (\sin^2 t + \cos^2 t)dt =$$

$$\int_{\pi}^{2\pi} 1 = \pi.$$

So already we know that this vector field is **not** independent of path.

Now I claim that it cannot be conservative either. Indeed, we get a closed path starting at P by taking the first path C_1 from P to Q and then the second path in the opposite orientation $-C_2$ from Q to P. Then

$$\int_{C_2 - C_1} F \cdot d\mathbf{r} = \int_{C_2} F \cdot d\mathbf{r} - \int_{C_1} F \cdot d\mathbf{r},$$

and we know this quantity is not zero because we just showed that the line integrals along the two different paths are not equal!

Conversely, suppose we had first found a closed path C around which the line integral of F was not zero – a good choice for this F would be the entire unit circle, oriented positively: we've seen that the line integral of F around the unit circle is 2π. Then we could choose any point Q different from the starting and ending point P of the closed path, and consider the closed path as being made up of C_1, the part from P to Q and $-C_2$, the rest of the path with the orientation reversed. These arguments are valid for any vector field, so we conclude:

A vector field is conservative if and only if it has the property that all line integrals are independent of path.

It might seem that no progress has been made – we now have two conditions that are equally impossible to verify directly, but in fact the path independence property can be used in an exciting way. Indeed, fix any point P_0 at which the vector field is defined. Then, to say that the vector field is independent of path is to say that for any other point Q, then the line integral of any path from P_0 to Q can be unambiguously written as

$$\int_{P_0}^{Q} F \cdot d\mathbf{r}.$$

But this reminds of a definite integral with a variable upper limit in one-variable calculus: if f is any (continuous) function, we can define a new function F by

$$F(x) = \int_{x_0}^{x} f(t)dt$$

and one version of the fundamental theorem of calculus is that F is an antiderivative of f: namely $F' = f$. In our case we have unfortunately already used the capital F for the vector field, but that is only a notational worry: we can *define* a scalar function

$$f(Q) = \int_{P_0}^{Q} F \cdot d\mathbf{r}.$$

Now comes the punchline: f really is an antiderivative of F, in the sense that $\nabla(f) = F$. We give the proof:

Write $F = Pdx + Qdy + Rdz$. We will show that $\frac{\partial f}{\partial x} = P$, and exactly the same method will give $\frac{\partial f}{\partial y} = Q$, $\frac{\partial f}{\partial z} = R$. By definition, the partial derivative of any function $f(x, y, z)$ is

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x, y, z) - f(x, y, z)}{\Delta x}.$$

So for our function we put $Q = (x, y, z)$ and look at

$$\frac{f(x + \Delta x, y, z) - f(x, y, z)}{\Delta x} = \frac{\int_{P_0}^{(x+\Delta x,y,z)} F \cdot d\mathbf{r} - \int_{P_0}^{(x,y,z)} F \cdot d\mathbf{r}}{\Delta x} =$$

$$\frac{\int_{(x,y,z)}^{(x+\Delta x,y,z,)} F \cdot d\mathbf{r}}{\Delta x} = \int_{(x,y,z)}^{(x+\Delta x,y,z)} Pdx,$$

since we can integrate over a path where y and z are both constant, hence $dy = dz = 0$. Now, by an averaging property of integrals, as we integrate a function about a smaller and smaller interval about a point and divide by the length of the interval, in the limit we will just get the value of the function at that point, so the limit as $\Delta x \to 0$ is just $P(x,y,z)$: i.e., $\frac{\partial f}{\partial x} = P$. In a similar way, we get $\frac{\partial f}{\partial y} = Q$ and $\frac{\partial f}{\partial z} = R$, so that indeed

$$\nabla(f) = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}) = (P,Q,R) = F.$$

To recap, we have shown that a vector field which is independent of path is the gradient of some function f. The converse is also true, and is easier: if $F = \nabla(f)$ is a gradient field, then F is independent of path, and indeed

$$\int_P^Q F \cdot d\mathbf{r} = f(Q) - f(P).$$

To see this, let $\mathbf{r}(t) = (x(t), y(t), z(t))$ be a parameterized curve with $\mathbf{r}(t_{\min}) = P$, $\mathbf{r}(t_{\max}) = Q$. Then

$$\int_C F \cdot d\mathbf{r} = \int_{t_{\min}}^{t_{\max}} F(x(t), y(t), z(t)) \cdot (x'(t), y'(t), z'(t))dt =$$

$$\int_{t_{\min}}^{t_{\max}} \frac{d}{dt} f(x(t), y(t), z(t))dt = f(x(t), y(t), z(t))|_{t_{\min}}^{t_{\max}} = f(Q) - f(P).$$

We sum up our findings in the following result, which is nothing less than a generalization of the Fundamental Theorem of Calculus for line integrals:

Theorem 1. *(Fundamental Theorem of Calculus for Line Integrals) Let F be a vector field defined in a region of the plane or in space. Then the following conditions on F are equivalent:*
a) F is conservative: for any closed curve $\int_C F \cdot d\mathbf{r} = 0$.
b) F is independent of path.
c) $F = \nabla(f)$ is a gradient vector field.
In case these conditions hold, the line integral of F along any path connecting P to Q is $\int_P^Q F \cdot d\mathbf{r} = f(Q) - f(P)$.

3. Potential functions and potential energy

Let us revisit Newton's vector field $F(x,y,z) = -(x,y,z)/||(x,y,z)||^3$ armed with the fundamental theorem of line integrals. Now we know that if we want to show that this vector field is conservative, we must find a function f such that $\nabla(f) = F$. We can take $f(x,y,z) = r^{-1}$, where $r = \sqrt{x^2+y^2+z^2}^{1/2}$. Indeed, $dr/dx = \frac{2x}{2r} = \frac{x}{r}$, and by symmetry $dr/dy = \frac{y}{r}$ and $dr/dz = \frac{z}{r}$. Thus $\frac{\partial f}{\partial x} = d(r^{-1})/dx = -r^{-2}dr/dx = -r^{-2}(x)/r = -x/r^3$. Similarly $\frac{\partial f}{\partial y} = -y/r^3$ and $\frac{\partial f}{\partial z} = -z/r^3$, which is what we wanted: $\nabla(f) = F$. So Newton's field *is* conservative, and equivalently it is independent of path: the work done along any path from initial point $P = (x_0, y_0, z_0)$ to final point $Q = (x_1, y_1, z_1)$ is

$$\int_P^Q F \cdot d\mathbf{r} = f(Q) - f(P) = \frac{1}{||(x_1,y_1,z_1)||} - \frac{1}{||(x_0,y_0,z_0)||}.$$

Put now

$$\varphi = -f(x, y, z) = \frac{-1}{||(x, y, z)||}.$$

Then φ, unlike f, is a function that is *increasing* with the distance to the origin r. Imagine we are piloting a rocket and we get it 1,000,000 km away from the origin: we have literally done a lot of work and invested a lot of energy to do this, energy that we could gain back by cutting the engines and letting the rocket fall back towards the earth. Thus we call φ the **potential function** for the conservative vector field F: it measures the **potential energy**.

Some comments are in order: first, you may notice that in this example, although φ increases with r, it is nevertheless always negative, just *less* negative the farther we get away from the origin. If this seems depressing, we should point out that energy is only well-defined as a relative quantity: that is, it is only *changes* in energy that are meaningful. Indeed, the change in potential energy $\varphi(Q) - \varphi(P)$ is $-\int_P^Q F \cdot d\mathbf{r}$ (since we do negative work to put energy into a system). Speaking in purely mathematical terms, we see that if we replaced φ by $\varphi + C$ for any constant C, we would get another potential function, since $\nabla(-f + C) = \nabla(f)$: it is still the case in this context that antiderivatives are well-defined only up to an additive constant, so it is a good thing that we subtract two values of the antiderivative in the fundamental theorem: $f(Q)+C-(f(P)+C) = f(Q)-f(P)$ does not depend on C.[1]

The equation $\varphi = C$ (or equivalently $f = -C$) consists of points which have the same potential energy C: accordingly, this is called an **equipotential surface**, and two points lie on the same equipotential surface precisely when there is no work done along any path between them. The path-independence of a gradient field can be seen as a generalization of the earlier fact that the gradient $\nabla(f)$ is always perpendicular to the level surface $f = C$ at any point: indeed, if the path stayed entirely in the $\varphi = C$ level surface, then the force would be perpendicular to the direction of motion at every point, so we are integrating the zero function. For Newton's vector field the level surfaces are spheres and we made this obervation at the beginning of the unit.

The term "conservative" refers to the fact that the change in potential energy around a closed path is zero for a conservative vector field. On the other hand, there is a more general law of conservation of **total mechanical energy** for a particle taking any kind of path in a conservative field. We keep track not just of the potential energy but also the **kinetic energy**, which is defined as $K = 1/2mv^2$, where v is the speed. Newton's second law reads $F = -\nabla\varphi = m\mathring{a} = m\mathbf{r}''$ or

$$m\frac{d\mathbf{v}}{dt} + \nabla\varphi = 0.$$

Dotting this quantity with the velocity $\mathbf{v} = d\mathbf{r}/dt$, we get the scalar equation

$$m\frac{d\mathbf{v}}{dt} \cdot \frac{d\mathbf{r}}{dt} + \nabla\varphi \cdot \frac{d\mathbf{r}}{dt} = 0. \quad (1)$$

[1]Conversely, if f_1 and f_2 are two functions on a connected region such that $\nabla(f_1) = \nabla(f_2)$, then $f_2 = f_1 + C$.

But note that $1/2mv^2 = 1/2m(\mathbf{v} \cdot \mathbf{v})$, so the derivative of the kinetic energy with respect to time is

$$d/dt1/2m(\mathbf{v}\cdot\mathbf{v}) = 1/2m(\frac{d\mathbf{v}}{dt}\cdot\mathbf{v} + \mathbf{v}\cdot\frac{d\mathbf{v}}{dt}) =$$

$$m(\frac{d\mathbf{v}}{dt}\cdot\mathbf{v}) = m(\frac{d\mathbf{v}}{dt}\cdot\frac{d\mathbf{r}}{dt}),$$

which is the first term in Equation (1). On the other hand, we know by the chain rule that

$$\nabla\varphi\cdot\frac{d\mathbf{r}}{dt} = d/dt(\varphi(\mathbf{r}(t)),$$

so Equation 1 is saying that the derivative of a certain quantity is zero:

$$d/dt(1/2mv^2 + \varphi(\mathbf{r}(t)) = 0. \tag{2}$$

Integrating this equation, we get that

$$1/2mv^2 + \varphi(\mathbf{r}(t)) = C,$$

i.e., that the sum of the kinetic and the potential energy is independent of time, exactly the conservation law we wanted.

4. Closed and exact differentials

Let us insist on a still more practical way of determining when a vector field is conservative. For instance, here are two vector fields in the plane:

$$F_1(x,y) = (2xy^3, 3y^2(x^2+1))$$

$$F_2(x,y) = (x^3+6y, 17x^2y^2).$$

We know either one will be conservative if and only if they are independent of path, if and only it is the gradient of some function. But how do we find the gradient?

Assume that $F = (P(x,y), Q(x,y)) = \nabla(f)$ is a gradient field, and we want to find f. This means that $\frac{\partial f}{\partial x} = P(x,y)$ and $\frac{\partial f}{\partial y} = Q(x,y)$. Integrating the first equation as a function of x we get $f(x) = p(x,y) + g(y)$, where $p(x,y)$ is a function such that $\frac{\partial}{\partial x}p(x,y) = P(x,y)$.[2] Note the undetermined function $g(y)$ – a function whose partial derivative with respect to x is a given function is determined only up to the addition of an arbitrary function of y. To learn what $g(y)$ is, we take the derivative with respect to y, getting

$$\frac{\partial f}{\partial y} = \frac{\partial p}{\partial y} + g'(y) = Q(x,y),$$

so that $g'(y) = Q(x,y) - \frac{\partial p}{\partial y}$, and integrating with respect to y, we'll get the expression for $g(y)$ and hence for f.

Let's do this for our first vector field: we integrate $P(x,y) = 2xy^3$ with respect to x, getting $f(x,y) = x^2y^3 + g(y)$. Now we differentiate with respect to y, getting $\frac{\partial f}{\partial y} = 3x^2y^2 + g'(y) = Q(x,y) = 3x^2y^2 + 3y^2$, so

$$g'(y) = 3x^2y^2 + 3y^2 - (3x^2y^2) = 3y^2.$$

[2] Again, we have used up our capital letters already for the vector field, so we use lowercase letters to denote antiderivatives.

No problem, $g(y) = y^3$, and we have discovered that

$$f(x, y) = x^2y^3 + y^3 \ (+C)$$

is such that $\nabla(f) = F$. Let's try this with F_2: $\frac{\partial f}{\partial x} = P(x, y) = x^3 + 6y$, so $f(x, y) = 1/4x^4 + 6xy + g(y)$, and $\frac{\partial f}{\partial y} = 6x + g'(y) = Q(x, y) = 17x^2y^2$. So we are supposed to have

$$g'(y) = 17x^2y^2 - 6x??$$

But this makes no sense: $g'(y)$ is supposed to only be a function of y! That is, we have discovered that F is not a gradient field at all, so it is not conservative.

There is a way to rephrase this procedure, as follows: if $F(x, y) = (P(x, y), Q(x, y)) = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$, then there is some relation between the two components $P(x, y)$ and $Q(x, y)$ of the vector field. Indeed, since $P = \frac{\partial f}{\partial x}$ and $Q = \frac{\partial f}{\partial y}$, we must have

$$\frac{\partial Q}{\partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial P}{\partial y}.$$

This condition is enough to ensure that the expression $Q(x, y) - \frac{\partial p}{\partial y}$ is only a function of y: indeed then

$$\frac{\partial}{\partial x}(Q(x, y) - \frac{\partial p}{\partial y}) = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0.$$

Thus before we use the above procedure to find the gradient, we should check whether the equality $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$ holds; one sometimes expresses this condition as saying that the "differential" $Pdx + Qdy$ is **closed** (we do not try to explain this terminology). On the other hand, note that the quantity $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ is just the scalar part of the **curl** of our vector field F, and we have rediscovered an earlier fact: $\text{curl}(\nabla f) = 0$, or in words, a gradient field is irrotational.

This has a very satisfying physical interpretation: suppose the curl of a vector field is *not* zero at a point. Then the vector field is *not* independent of path, i.e., energy is not being conserved. But recall that our (so far unjustified) physical interpretation of the curl is that if we nailed down a paddlewheel at that point, oriented in an appropriate direction, then the paddlewheel will rotate. And what is it that makes the paddlewheel rotate? Energy! The fact that in this circumstance energy is *lost* from the system (and gained by us) is the source of hydroelectric power!

Although we used planar vector fields as examples, an analogous discussion is valid for vector fields $F = (P, Q, R)$ in space: gradient vector fields are irrotational. Indeed, writing down $\frac{\partial f}{\partial x} = P$, $\frac{\partial f}{\partial y} = Q$ and $\frac{\partial f}{\partial z}$ and taking all mixed second order partials leads to the equations $\frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}$, $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, $\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$, which is equivalent to $\text{curl}(F) = 0$. And similarly, if one has an irrotational vector field, one can apply a procedure of repeated integration and differentiation to try to find the gradient: let $f_1(x, y, z)$ be an antiderivative of P with respect to x, $f_2(y, z) := Q - \frac{\partial}{\partial y} f_1$, $f_3(y, z)$ be an antiderivative of $f_2(y, z)$ with respect to y, $f_4(z) := R - \frac{\partial}{\partial z} f_3$ and $f_5(z)$ be an antiderivative of f_4 with respect to z. Then

$$f = f_1(x, y, z) + f_2(y, z) + f_3(z) + C.$$

For an example of this process, see pages 500-501 of your textbook. (Truthfully it is a bit of a pain and will not come up much.)

5. Not all irrotational vector fields are gradient fields

We noted in the last section that every gradient field is irrotational. Conversely, if we have an irrotational vector field (zero curl), we gave an explicit procedure for reconstructing the gradient. So how can it be that not all irrotational fields are gradient fields??

The first answer is that seeing is believing: recall our "very special" vector field $F_\star(x, y) = (\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2})$. We computed in Handout 5 that $\text{curl}(F_\star) = 0$ and in Handout 6 that the line integral $\int_C F_\star = 2\pi$, where C is any circle centered at the origin: not conservative! Indeed, recall that the flow lines of this vector field are counterclockwise circles, so visibly the work done in going around a circle is positive.

Let's apply our procedure to find the gradient of such a field: it will be slightly cleaner to reverse the order and first find a function $q(x, y)$ such that $\frac{\partial}{\partial y} q = Q(x, y) = \frac{x}{x^2+y^2}$. To do this integral, recall that

$$\int \frac{dt}{a^2 + t^2} = (1/a) \arctan(t/a).$$

So an antiderivative of Q with respect to y is x times an antiderivative of $\frac{1}{x^2+y^2}$ with respect to y or $q(x, y) = x(1/x) \arctan(y/x) = \arctan(y/x)$. Now we are prepared to modify q by a function of $g(x)$ so that $\frac{\partial}{\partial x} q = P$, but in fact this is not necessary: already we have

$$\frac{\partial}{\partial x} q = \frac{\partial}{\partial x} \arctan(y/x) = \frac{y \frac{\partial}{\partial x}(1/x)}{1 + (y/x)^2} =$$

$$\frac{-y/x^2}{1 + y^2/x^2} = \frac{-y}{x^2 + y^2} = P(x, y).$$

So the plot thickens because we have found a gradient function, namely $f(x, y) = \arctan(y/x)$. On the other hand, this function is not defined along the y-axis, when the denominator vanishes. So if we took a closed path to the right of the line $x = 0$, then the line integral would indeed be zero, and similarly if we took a path to the left of $x = 0$, but our circular path crosses the line, and at this point the gradient is not defined. In fact $\arctan(y/x)$ is a very familiar function: it is just θ, the angular coordinate of the point (x, y). So the problem doesn't really have to do with the vertical line $x = 0$, but rather the fact that θ really is *not* a continuous function along closed paths which wind around the origin: if we start at angle 0, then by the time we come full circle θ approaches 2π: in order to insist that the angular coordinate be a well-defined function with values in $[0, 2\pi)$ (say), we must resign ourselves to the fact that as we cross the positive x-axis theta jumps discontinuously from a value very close to 2π to a value very close to zero. In differential form, then, our line integral takes the form

$$\int_0^{2\pi} d\theta = 2\pi.$$

In fact line integrals over the vector field $F_\star$ are "almost" conservative in the sense that the line integral of $F_\star$ along any closed curve is $2\pi n$, where n is the **winding**

number, the number of times the path winds around the origin in a counterclockwise direction.

The details of this example are rather intricate but the moral is this: if the curl of a vector field at a point is zero, then the line integral is the gradient of a function f defined **locally** near that point. However the function f may or may not extend to the entire domain of definition of the vector field. This is an instance of the fact of life that one is often guaranteed that differential equations have solutions locally about some point (an initial condition), but there is no guarantee that the solutions will exist for all time.

Here is a simpler example of this: consider the differential equation $dy/dt = y^2$ with initial condition $y(0) = 1$. What is the limit of the solution curve $y = y(t)$ as $t \to \infty$? This is a separable differential equation, so $dy/y^2 = dt$; integrating we get $-1/y = t + C$, or $y = \frac{-1}{t+C}$. Plugging in $t = 0$ we want $1 = y(0) = \frac{-1}{0+C} = \frac{-1}{C}$, so $C = -1$. Thus we have the solution curve $y = \frac{-1}{t-1}$. Now I'm sorry to tell you that this function doesn't do *anything* as $t \to \infty$, as it already blows up (approaches positive infinity) at time $t = 1$. The **global** behavior of differential equations is always a more delicate process and moreover is sensitive to the qualitative features – what mathematicians call the **topology** – of the domain.

Recall that our vector field $F_\star$ has a singularity at the origin – a "hole" – and we saw that the line integral of $F_\star$ around a closed path is not conservative only by virtue of its awareness of the number of times it goes around the hole. This makes us suspect that if we integrated an irrotational vector field around a simple closed curve in whose interior there were no "holes" in the vector field, then we would in fact get zero. This is true and will be discussed as an application of Green's Theorem.