HANDOUT SIX: LINE INTEGRALS

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1. Introduction to Line Integrals: the need for work

One of the most fundamental concepts of physics is the notion of **energy**, defined (enigmatically enough, as usual in the subject) as "the ability to do work." Work in turn is defined as the quantity that results when a force is exerted over a distance. This notion, however, admits descriptions at various levels of sophistication, as we now recall:

If the force is constant and our motion is in a straight line, then indeed the work is defined as just the product of the force times the distance: if we move a 100 pound weight over a distance of 10 feet, then indeed the work done is 1000 foot pounds. This is the presentation typically given in a high school physics class.

But this is not real life! Suppose next that we are still moving in a straight line but the force is variable: indeed, the typical example of this is that of a **spring**: suppose we have a mass M on a spring, and we stretch the mass a distance of x units from its equilibrium position. The spring will pull back in the opposite direction, and the early British scientist Robert Hooke formulated a simple rule describing the force: it is simply proportional to the distance: $F(x) = -kx$, where the k is called the *spring constant* since it emphasizes the strength of the spring.[1] Now the definition of work you would learn in a freshman physics class is that

$$W = \int F dx.$$

So in our example, the work required to move a spring d units from equilibrium position is $\int_0^d -kx = -1/2kx^2|_0^d = -1/2kd^2$. Note the minus sign, which has an important physical meaning: if we move *opposite* the direction of the force, negative work is done, whereas if we move in the direction of the force, positive work is done. This makes sense in terms of energy: the minus sign means we must exert energy to stretch the spring.

As a quick aside, I cannot resist mentioning that this is also a good example of Newton's Second Law: $F = ma$. Since $a = x''$ and $F = -kx$, this gives $-kx = mx''$, or

$$x'' = -k/mx.$$

[1]By the way, as engineers you doubtless know to be suitably wary of the fact that this simple law must at best be an approximation which is reasonable only under certain conditions: suppose that your Slinky pulls back with one pound of force when stretched one foot. Then Hooke's Law says that if we stretch the Slinky one mile it will pull back with a force of 5280 pounds, which is enough to move your car. I hope you don't actually believe this!

This is a second-order differential equation, whose general solution is

$$x(t) = A\cos(\omega t) + B\sin(\omega t),$$

where $\omega = \sqrt{\frac{k}{m}}$. That is, Newton's Law implies that the motion of a particle on the end of a spring will be **periodic** with period $\frac{2\pi}{\omega}$, i.e., we get the so-called **simple harmonic motion**.

However even one-variable calculus is not enough to really understand the notion of work, because we are missing a key element: it is only the **tangential** component of the force that counts, i.e., only the force in the direction of the motion. In other words, if a particle moves in such a way so that its only acceleration is normal acceleration – the best example being a particle moving in a circle with constant speed! – then the work is zero. This makes sense, since after a particle travels around in a complete circle no energy is gained or lost.

If the particle happens to be moving in a straight line, and the force is given by a vector field $F(x, y, z)$, then we can still write down an expression for the work using the tools we already have: indeed, say the particle is moving from initial point (x_0, y_0, z_0) to final point (x_1, y_1, z_1); we parameterize this as

$$L(t) = (x_0, y_0, z_0)+t(x_1-x_0, y_1-y_0, z_1-z_0) = (x_0+t(x_1-x_0), y_0+t(y_1-y_0), z_0+t(z_1-z_0).$$

Note that $L(0) = (x_0, y_0, z_0)$ and $L(1) = (x_1, y_1, z_1)$, and the direction of the line is $\mathbf{v} := (x_1 - x_0, y_1 - y_0, z_1 - z_0)$. Then the following integral gives the work:

$$W = \int_0^1 F(L(t)) \cdot \mathbf{v} dt.$$

On the other hand, if the particle is moving along a *curved path* – like a circle – then we need some way to integrate the dot product of the force and the velocity vector *along the curve*. That is, we need to be able to integrate a function defined on a curved line, which brings us to the notion of a **line integral**.

2. Line integrals of scalar functions

We begin by figuring out how to integrate a *scalar* function over a curve. For instance, suppose C is a curve in the plane or in space, and $\rho(x, y, z)$ is a function defined on C, which we view as a **density**. For example, imagine C is a thin wire and $\rho(x, y, z)$ gives the mass density of the wire at the point (x, y, z). To get the total mass of the wire we should perform some integral $\int_C \rho$, whatever that means. To get a hint, imagine the density is identically equal to one. Recalling that density is mass over volume, if the density is one, the mass should be the volume. Since the "volume" of a one-dimensional object is its length, the hint that we get in trying to define the line integral is that $\int_C 1$ should be the **arc length** of C.

Aha – we already know how to compute the arclength! Let $\mathbf{r}(t) = (x(t), y(t), z(t))$ be a parameterization of the curve. Then the speed of the curve is $v = ||\mathbf{v}(t)|| = \sqrt{(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2}$ and the arclength is the integral of the speed, so $\int_{t_{\min}}^{t_{\max}} v dt = \int_{t_{\min}}^{t_{\max}} \sqrt{(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2}$, where $\mathbf{r}(t_{\min})$ and $\mathbf{r}(t_{\max})$ are the initial and final points of the curve.

If this is how we integrate 1 on the curve, one can guess that the correct definition of $\int_C \rho$ is

$$\int_{t_{\min}}^{t_{\max}} \rho(\mathbf{r}(t))v(t)dt = \int_{t_{\min}}^{t_{\max}} \rho(\mathbf{r}(t))\sqrt{x'^2 + y'^2 + z'^2}dt.$$

This is indeed correct; let's check it by a Riemann sum calculation. Indeed, if we want to integrate the function ρ over a straight line $L(t) = (x_0, y_0, z_0) + \mathbf{v}t$, then $L'(t) = \mathbf{v}$, a constant function, and the speed is just $v = ||\mathbf{v}|| = \sqrt{v_x^2 + v_y^2 + v_z^2}$. As mentioned above, we know how to do the integral in this case: it would just be $\int_{t_{\min}}^{t_{\max}} F(\mathbf{r}(t))vdt$, and this integral itself comes from dividing the interval $[t_{min}, t_{max}]$ up into very small subintervals Δt and approximating the function as being constant on each of these subintervals.

In the general case, we will approximate $\rho(\mathbf{r}(t))$ by a constant function *and* approximate the path $\mathbf{r}(t)$ by a polygonal path: i.e., we will approximate the velocity on each subinterval Δt_i as being a constant. Suppose we divide the entire interval $[t_{\min}, t_{\max}]$ up into n pieces, and on each piece we approximate the speed by

$$\sqrt{(\frac{\Delta x_i}{\Delta t_i})^2 + (\frac{\Delta y_i}{\Delta t_i})^2 + (\frac{\Delta z_i}{\Delta t_i})^2}.$$

Since the speed is the derivative of arclength with respect to time, we can write this as $\frac{\Delta s_i}{\Delta t_i}$, i.e., we put

$$\Delta s_i := \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2 + (\Delta z_i)^2},$$

and regard Δs_i as a small change in arclength. Having made these approximations, we form the Riemann sum

$$\sum_{i=1}^{n} \rho(\mathbf{r}(t_i^\star)) \left(\frac{\Delta s_i}{\Delta t_i}\right) \Delta t_i,$$

where $t_i^\star$ is some value in the subinterval $[t_{i_1}, t_i]$. Now as the length of each subinterval goes to zero, $\left(\frac{\Delta s_i}{\Delta t_i}\right) \to ds/dt = v(t) = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$, and the sum becomes

$$\int_{t_{\min}}^{t_{\max}} \rho(\mathbf{r}(t))v(t)dt = \int_{t_{\min}}^{t_{\max}} \rho(\mathbf{r}(t))(ds/dt)dt.$$

We may abbreviate $ds = (ds/dt)dt$ and we arrive at the definition of the **scalar line integral**

$$\int_C \rho \; ds = \int_{t_{\min}}^{t_{\max}} \rho(\mathbf{r}(t))||\mathbf{r}'(t)||dt.$$

Again, note that if $\rho \equiv 1$, then we will just be integrating the speed to get the arclength; this is a check that the preceding Riemann sum argument is reasonable.

Example: Find the line integral of the function $\rho(x, y) = xy$ on the unit circle.

Solution: We use the parameterization $\mathbf{r}(t) = (\cos t, \sin t)$, with $t_{\min} = 0$, $t_{\max} = 2\pi$. Note that $\mathbf{r}'(t) = (-\sin t, \cos t)$, so $v(t) = ||\mathbf{r}'(t)|| = 1$: this parameterization has unit speed. Thus the line integral is

$$\int_{t_{\min}}^{t_{\max}} \rho(\mathbf{r}(t))||\mathbf{r}'(t)||dt =$$

$$\int_0^{2\pi} (\cos t \sin t)dt = \int_0^{2\pi} 1/2\sin(2t)dt =$$

$$-1/4\cos(2t)|_0^{2\pi} = -1/4(\cos(2\pi) - \cos(0)) = -1/4(1-1) = 0.$$

In hindsight we could have seen this geometrically by noting that the contribution of the line integral from the first and third quadrants will be positive, the contribution from the second and fourth quadrants will be negative, and these two contributions will cancel each other precisely.

Now we must admit that there is a technical point that we slid under the rug: our definition of a line integral of ρ along C used a particular parameterization of C, whereas in the example we just said "take the line integral along the unit circle." Indeed we want the line integral to be – like the curvature – a function which is independent of the chosen parameterization of the curve: for instance, if we are interpreting it as the mass of a wire with density function $\rho(x, y, z)$, then of course the mass should be independent of the parameterization.[2] We will not prove this independence of parameterization in class, but it is recorded in an appendix.

3. Line integrals over vector fields: work

We now return to the problem that motivated the discussion: suppose a particle moves through a field of forces F via path C. The work done is supposed to be the integral of the tangential component of the force over the curve. This now makes sense in terms of line integrals, as follows: let $\mathbf{r}(t) = (x(t), y(t), z(t))$ be a parameterization of C, so that $\mathbf{v}(t) = \mathbf{r}'(t)$ is the velocity vector. Then at any time t, the tangential component of the force is $F(\mathbf{r}(t)) \cdot T(t)$, where $T(t) = \mathbf{v}(t)/v$ is the unit tangent at time t. (We assume as usual that the curve has nonzero speed at every point.) Thus the work done is $\int_{t_{\min}}^{t_{\max}} F(\mathbf{r}(t)) \cdot T(t)v(t)dt$. But since the unit tangent times the speed is precisely the velocity vector $\mathbf{v}(t)$, this expression simplifies to

$$W = \int_{t_{\min}}^{t_{\max}} F(\mathbf{r}(t)) \cdot \mathbf{v}(t)dt = \int_C F(\mathbf{r}) \cdot d\mathbf{r}.$$

We note that because this is a special case of a line integral of a scalar function, this too is independent of the parameterization. However, it is **NOT** independent of the **orientation** of the curve: since we are taking $F \cdot T(t)$, if we traversed the path in the opposite direction, $T(t)$ would be multiplied by -1, hence the entire line integral would be multiplied by -1. This makes sense physically: since work represents a change in energy, if we reverse time, what was formerly a gain in energy will now be a loss of energy, and vice versa.

[2] Imagine the merchandise of a shop is stored in a winding tunnel underneath the shop. The shop is visited by a conscientious insurance company, who wants to record for itself the total value of the merchandise. The company is so conscientious that it hires two different surveyors to perform this job. If the two surveyors take inventory at two different rates, one dawdling at certain points and the other speeding through, then the functions " dollars of merchandise recorded per second" will be completely different for these two workers, but in the end the total dollar amount will be the same.

Finally we introduce a piece of notation: if $F(x, y, z) = (P(x, y, z), Q(x, y, z), R(x, y, z))$ is a vector field and $\mathbf{r}(t) = (x(t), y(t), z(t))$ is a path, then the line integral $\int_C F(\mathbf{r}) \cdot d\mathbf{r}$ comes out as

$$\int_{t_{\min}}^{t_{\max}} (P(x(t), y(t), z(t)), Q(x(t), y(t), z(t)), R(x(t), y(t), z(t)) \cdot (x'(t), y'(t), z'(t)) dt =$$

$$\int_{t_{\min}}^{t_{\max}} (P(x(t), y(t), z(t)) \frac{dx}{dt} + Q(x(t), y(t), z(t)) \frac{dy}{dt} + R(x(t), y(t), z(t)) \frac{dz}{dt}) dt.$$

By change of variables it is acceptable – and more efficient – to write this as

$$\int_{t_{\min}}^{t_{\max}} P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz.$$

One sometimes says of the above expression that the line integral is written in **differential form**.

Let us now look at some examples of line integrals; we will use the same vector fields we considered in Handout 5.

Example: Let $F = -F_a = (-x/r^a, -y/r^a)$, where a is some number and $r = \sqrt{x^2 + y^2}$: recall that this vector field points radially outward at every point. We will compute the line integral of F_a around the circle centered at the origin and of radius r. But before we do we will predict the answer! Indeed, since this vector field points radially outward and our path is circular, we have that the force is always perpendicular to the direction of motion, so the work done had better be zero. Indeed, since $||(-x/r^a, -y/r^a)|| = r^{-a}||(-x, -y)|| = r^{1-a}$, when $a = 3$ this is just the planar form of Newton's inverse square law, so we are confirming the fact that a planet in circular motion neither gains nor loses energy.

Okay, let's do it: $\mathbf{r}(t) = (r \cos t, r \sin t)$, so $\mathbf{v}(t) = (-r \sin t, r \cos t)$, so

$$F(\mathbf{r}) \cdot \mathbf{v} = (-r/r^a \cos t, -r/r^a \sin t) \cdot (-r \sin t, r \cos t) =$$
$$r^2/r^a \sin t \cos t - r^2/r^a \sin t \cos t = 0,$$

so $W = \int_0^{2\pi} 0 = 0$, which is a relief.

By the way, planetary orbits are elliptical, and not necessarily circular. In particular, Kepler's second law implies that the tangential component of the acceleration is usually *not* zero in an elliptical orbit, so that the force is *not* usually perpendicular to the direction of motion, and the function $F \cdot \mathbf{v}$ is not identically zero. Still, we feel that since the orbits are periodic, there can be no loss of energy: after all, one does not need to "fuel" the planets' motion around the sun. We will see in the next section that this vector field F_a has a certain property implying that line integrals around *all* closed curves are zero. Such vector fields are called **conservative** for precisely this region: energy is conserved.

Example: The same discussion holds for the vector field $F_a = (x/r^a, y/r^a)$, since F_a is perpdendicular to the velocity at any point. (If $F \cdot v = 0$, then $(-F) \cdot v = 0$.) But now consider the vector field $G_a = L(F_a) = (-y/r^a, x/r^a)$ obtained by turning every vector in F_a 90 degrees to the left. Now the flow lines of the vector field are counterclockwise circles, so if we integrate the vector field around a counterclockwise circle, there must be positive work done: in other words, F and $\mathbf{v}$ are *parallel* at every point, so there will be positive work done, a gain of energy.

Indeed, $F(\mathbf{r}(t)) = (-r/r^a \sin t, r/r^a \cos t)$, while $\mathbf{v}(t) = (-r \sin t, r \cos t)$, so

$$W = \int_0^{2\pi} F(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt = \int_0^{2\pi} (-r/r^a \sin t, r/r^a \cos t) \cdot (-r \sin t, r \cos t) dt =$$

$$\int_0^{2\pi} r^2/r^a (\sin^2 t + \cos^2 t) dt = \int_0^{2\pi} r^2/r^a = 2\pi r^{2-a} > 0.$$

Note that the amount of work done depends on a as follows: if $a < 2$, then r^{2-a} is a positive power of r, and as we allow $r \to 0$ the amount of work done approaches zero. (When $a < 0$, the vector field extends continuously to the origin, and the fact that this limit is zero is automatic: if we integrate a bounded function over a curve whose arclength goes to zero, the line integral must also go to zero. However when $a \geq 0$ the vector field F_a has a singularity at the origin, and when $a > 1$ you would not want to be swimming near the origin: the speed of the current approaches infinity!) When $a > 2$ $\lim_{r \to 0} r^{2-a} \to +\infty$. However, at the special value $a = 2$ the work is just 2π no matter what r is. Note when $a = 2$ $F_a = F_\star = \frac{-y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy$ is our "very special vector field" from Handout 5. In particular $\mathrm{curl}(F_a) = 0$.

4. Normal line integrals in the plane

If $\mathbf{r}(t)$ is a parameterized **plane curve** and F is a vector field in the plane, then instead of integrating the dot product of F and the velocity vector $\mathbf{v}$, we can integrate the dot product of F and the vector $R(\mathbf{v})$ which is 90 degrees to the right of the normal vector. If C is a simple closed curve oriented counterclockwise, the **normal line integral**

$$\int_C F(\mathbf{r}(t)) \cdot R(\mathbf{v}(t)) dt$$

measures the **total flux** through the boundary of the region R enclosed by C. On the other hand, we showed in Handout 5 that if C is a rectangle, the divergence is equal to the flux density, so that the total flux is the integral of the divergence. If the notion of divergence has any physical meaning, then no matter what the shape of the region is, the double integral of the flux density should give the divergence, i.e., we should have the following equation:

$$\int\int_R \mathrm{Div}(F) dx dy = \int_C F \cdot R(\mathbf{v}(t)) dt. \tag{1}$$

This equation is indeed true and is one formulation of one of the most important results of the course, **Green's Theorem**. We will discuss Green's Theorem in about a week, and one of the things we will do is see why the "turning operator" business of Handout 5 implies that we get another form of Green's Theorem equating the double integral of the **curl** of the vector field F on R to the usual (tangent) line integral of F on the boundary of R.

5. Appendix: Invariance of line integrals under reparameterization

Let $\mathbf{r}(t) = (x(t), y(t), z(t))$ be a parameterized curve, let $\rho(x, y, z)$ be a scalar function defined on the curve, so that

$$I_1 = \int_C \rho = \int_{t_{\min}}^{t_{\max}} \rho(\mathbf{r}(t))||\mathbf{r}'(t)||dt$$

gives the line integral of ρ along C using the parameterization $\mathbf{r}$.

Let $t \mapsto u(t)$ be a change of parameterization: we assume that $u'(t) > 0$ for all t (i.e., that u is an increasing function of t).[3] We want to show that the line integral using the parameterization $\mathbf{r}(u(t))$ is the same as the above line integral. That is, with the new upper and lower limits $u^{-1}(t_{\min})$ and $u^{-1}(t_{\max})$, we have a second expression

$$I_2 = \int_{u^{-1}(t_{\min})}^{u^{-1}(t_{\max})} \rho(x(u(t)), y(u(t)), z(u(t)))||d(r \circ u)/dt||dt,$$

and we want to show that $I_2 = I_1$. As in all such matters, this comes down to a chain rule calculation: here we use that $d(r \circ u)/dt = (dr/du)(du/dt)$. Taking norms, and recalling that $du/dt > 0$, we have $||d(r \circ u)/dt|| = ||dr/du||du/dt$, so

$$I_2 = \int_{u^{-1}(t_{\min})}^{u^{-1}(t_{\max})} \rho(x(u), y(u), z(u))||dr/du||du/dtdt.$$

But now, performing the substitution $u = u(t)$ on this definite integral and changing the limits to u-limits – this is an application of the formula

$$\int_a^b f(u)du/dxdx = \int_{u(a)}^{u(b)} f(u)du,$$

we get that

$$I_2 = \int_{t_{\min}}^{t_{\max}} \rho(x(u), y(u), z(u))||dr/du||du = I_1,$$

as we wanted to show.

Note the importance of the factor of $||\mathbf{r}'(t)||$ in the definition of the scalar line integral – it is what makes the integral independent of the choice of parameterization. If you like to think in terms of dimensional analysis, it also needs to be there for dimensional reasons: suppose we are integrating a linear mass density function ρ, so the units of ρ are (say) kilograms per meter. The units of the integral $\int_{t_{\min}}^{t_{\max}} \rho(\mathbf{r}(t))dt$ are kilogram seconds per meter – this is not a unit of mass! Putting in the $||\mathbf{r}'(t)||$ corrects this by multiplying by meters per second, so that the units become kilogram seconds per meter times meters per second, or kilograms, as it should be.

[3] The result remains true if $u'(t) < 0$ for all t.