QUADRATIC FORMS OVER DISCRETE VALUATION FIELDS

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A ring R is **non-dyadic** if $2 \in R^\times$; otherwise R is **dyadic**.

As a general rule, the theory of quadratic forms over a ring R goes more smoothly if R is non-dyadic. Of course, if R is a field then this simply says that we wish to avoid characteristic 2, but in general there is more to it than this. For instance, the ring $\mathbb{Z}$ is dyadic according our definition whereas for an odd prime power q, the ring $\mathbb{F}_q[t]$ is not, and indeed the theory of quadratic forms is somewhat easier over the latter ring than the former. Of course if R is a dyadic domain of characteristic different from 2 then $R[\frac{1}{2}]$ is nondyadic.

The above dichotomy becomes especially clear when we consider the case of a **CDVR**, i.e., a ring which is complete with respect to a discrete valuation v. Our primary goal here is the analysis of quadratic forms over the *fraction field* F of a CDVR, a complete discretely valued field. (In other words, we are treating the *local algebraic theory* of quadratic forms. The *local arithmetic theory* is to a number theorist even more important, but it is richer and will have to be treated later.) Nevertheless we will certainly see the ring R play a role in our analysis.

1. Non-dyadic CDVFs

Let R be a CDVR with fraction field K, valuation $v : K^\times \to \mathbb{Z}$, maximal ideal $\mathfrak{p}$, residue field $k = R/\mathfrak{p}$.

Lemma 1. *Let R be Henselian and nondyadic. Then for $x \in R^\times$, TFAE:*
(i) x is a square in $K^\times$.
(ii) x is a square in $R^\times$.
(iii) The image $\overline{x}$ of x in k is a square in $k^\times$.

Proof. (i) $\implies$ (ii): Suppose there exists $y \in K$ with $y^2 = x$. Then y satisfies the monic polynomial equation $t^2 - x = 0$ so is integral over R. But R, being a DVR, is integrally closed, so $y \in R$. Moreover, for any elements x, y in a *commutative* ring R, $xy \in R^\times \iff x, y \in R^\times$, so $y^2 \in R^\times \implies y \in R^\times$.
(ii) $\implies$ (iii): Indeed, if there is $y \in R^\times$ with $y^2 = x$, then after applying the quotient map we have $\overline{y}^2 = \overline{x}$.
(iii) $\implies$ (i): Let $f(t) = t^2 - x$, let $\overline{y} \in k$ be such that $\overline{y}^2 = \overline{x}$, and let $\tilde{y}$ be any lift of $\overline{y}$ to R. Since $\overline{f(\tilde{y})} = 0$, $|f(\tilde{y})| < 1$. Since R is nondyadic, $\overline{f'(\tilde{y})} = 2\overline{y} \neq 0$, so $|f'(\tilde{y})| = 1$. So $|f(\tilde{y})| < |f'(\tilde{y})|^2$, and Hensel's Lemma gives a root of f. □

We immediately deduce the following key result.

Corollary 2. *Let R be a Henselian nondyadic DVR. Then the canonical map $R \to k$ induces an isomorphism of groups $r : R^\times/R^{\times 2} \to k^\times/k^{\times 2}$.*

Remark 3. *The conclusion of Corollary 2 is the only completeness property of R that will be needed for the coming results. So we could axiomatize this result by calling a non-dyadic DVR **quadratically Henselian** if the natural map $r : R^\times/R^{\times 2} \to k^\times/k^{\times 2}$ is an isomorphism. We have nothing specific to gain from this, so we will not use this terminology, but see e.g.* [S, p. 208].

Lemma 4. *Let R be a DVR.*
a) There is a short exact sequence

$$1 \to R^\times \to K^\times \xrightarrow{v} \mathbb{Z} \to 0.$$

This sequence is split, and splittings correspond to choices of a uniformizer π.
b) If R is Henselian and nondyadic, then there is a split exact sequence

$$1 \to k^\times/k^{\times 2} \to K^\times/K^{\times 2} \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

Proof. Part a) is immediate. Modding out by squares, we get a split short exact sequence

$$1 \to R^\times/R^{\times 2} \to K^\times/K^{\times 2} \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

Further assuming that R is nondyadic and Henselian, we use the isomorphism of Corollary 2 to get the desired result. □

In particular, any nondegenerate n-ary quadratic form has a diagonal representation such that each coefficient has valuation 0 or 1 and thus a representaiton of the form

$$(1) \qquad q(x,y) = u_1 x_1^2 + \ldots + u_r x_r^2 + \pi v_1 y_1^2 + \ldots + \pi v_s y_s^2 = q_1(x) + \pi q_2(y),$$

with $u_i, v_j \in R^\times$ and $r + s = n$.

Theorem 5. *Let R be a nondyadic DVR with fraction field K, uniformizer π and residue field k. Let $n \in \mathbb{Z}^+$ and let $r, s \in \mathbb{N}$ with $r+s = n$. Let $u_1, \ldots, u_r, v_1, \ldots, v_s \in R^\times$, and let*

$$(2) \qquad q(x,y) = u_1 x_1^2 + \ldots + u_r x_r^2 + \pi v_1 y_1^2 + \ldots + \pi v_s y_s^2 = q_1(x) + \pi q_2(y),$$

be an n-ary quadratic form. Also write $\overline{q_1}$ and $\overline{q_2}$ for the reductions of q_1 and q_2 modulo π: these are nondegenerate quadratic forms over k.

a) Suppose that $\overline{q_1}$ *and* $\overline{q_2}$ *are anisotropic over* k*. Then* q *is anisotropic over* K*.*
b) Suppose that K *is Henselian and* q *is anisotropic over* K*. Then* $\overline{q_1}$ *and* $\overline{q_2}$ *are both anisotropic over* k*.*

Proof. a) Suppose $\overline{q_1}$ and $\overline{q_2}$ are both anisotropic over k and, seeking a contradiction, that q is isotropic. By rescaling, we get a *primitive* vector (x, y) such that $q(x, y) = 0$: that is, all x_i, y_j lie in R and not all of them are divisible by π.
Case 1: Suppose there exists $1 \leq i \leq r$ such that $v(x_i) = 0$. Then reducing the equation $q(x, y) = 0$ modulo $\mathfrak{p}$ gives $\overline{q}(x, y) = \overline{q_1}(\overline{x}) = 0$. Since $\overline{x_i} \neq 0$, $\overline{q_1}$ is isotropic over k, a contradiction.
Case 2: Suppose $\pi \mid x_i$ for all $1 \leq i \leq r$ and $v(y_j) = 0$ for some $1 \leq j \leq s$. Then $\pi^2 \mid q_1(x)$, so the equation $q_1(x) + \pi q_2(y) = 0$ implies $\overline{q_2}(\overline{y}) = 0$. Since $\overline{y_j} \neq 0$, $\overline{q_2}$ is isotropic over k, a contradiction.
b) Suppose $\overline{q_1}$ and $\overline{q}_2$ are *not* both anisotropic over k. If $\overline{q_1}$ is isotropic over k, there is $\overline{x} \in k^r$ with $\overline{q_1}(\overline{x}) = 0$ and such that $\overline{x_i} \neq 0$ for at least one i. Then $\frac{\partial \overline{q}_1}{\partial x_i} = 2x_i \neq 0 \in k$, so by Hensel's Lemma there is $x' \in R^r$ such that $x' \pmod{\pi} = \overline{x}$ and $q_1(x') = 0$. In particular $x'_i \neq 0$, so q_1 is isotropic over K. Since q_1 is a subform of q, also q is isotropic over K. Similarly, if $\overline{q_2}$ is isotropic over k then q_2 is isotropic over K and thus so is the subform πq_2 of q, so q is isotropic over K. □

Corollary 6. *For* R *a non-dyadic DVR with fraction field* K *and residue field* k*:*
a) We have $u(K) \geq 2u(k)$*.*
b) If R *is Henselian, then* $u(K) = 2u(k)$*.*

Proof. a) Let $\overline{q}$ be an anistropic n-ary quadratic form over k, and let q be any lift of $\overline{q}$ to a quadratic form with R-coefficients. Then by Theorem 5a) $q(x, y) = q(x) + \pi q(y)$ is anisotropic over K. Thus $u(K) \geq 2u(k)$.
b) By Lemma 5, every nondegenerate n-ary quadratic form is K-equivalent to a form q as in (2). So if K is Henselian and $n > 2u(k)$, then $\max r, s > n$ so at least one of $\overline{q_1}, \overline{q_2}$ is isotropic over k. By Theorem 6b), q is isotropic. □

Corollary 7. *Let* K *be a CDVF with residue field* $\mathbb{F}_q$*,* q *odd.*
a) We have $u(K) = 4$*.*
b) Let $r \in \mathbb{F}_q^\times \setminus \mathbb{F}_q^{\times 2}$*. Then an explicit anisotropic quaternary form over* K *is*

$$q(x, y, z, w) = x^2 - ry^2 + \pi z^2 - \pi r w^2.$$

Proof. Exercise. □

By looking more carefully at we have already done, we get the following result.

Theorem 8. *(Springer) Let* R *be a nondyadic Henselian DVR with fraction field* K *and residue field* k*. The map* $q \mapsto (\overline{q_1}, \overline{q_2})$ *induces an isomorphism of Witt groups*

$$\delta : W(K) \xrightarrow{\sim} W(k) \oplus W(k).$$

1.1. $\mathbb{C}((t))$**.**

Let $\mathbb{C}$ be a field of characteristic different from 2 which is quadratically closed: $\mathbb{C}^\times = \mathbb{C}^{\times 2}$. We will classify quadratic forms over the field $\mathbb{C}((t))$. Indeed, this is easy. By Springer's Theorem and Proposition II.11 we have a group isomorphism

$$W(\mathbb{C}((t)) \cong W(\mathbb{C}) \times W(\mathbb{C}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}. \quad (3)$$

The three nontrivial anisotropic quadratic forms are

$$\langle 1 \rangle, \langle t \rangle, \langle 1, t \rangle.$$

By Corollary 6, we have

$$u(\mathbb{C}((t)) = 2.$$

Thus in the terminology of §II.2, $\mathbb{C}((t))$ is quadratically C^1: every binary form is universal. By Theorem II.13, since $-1 \in \mathbb{C}^{\times 2}$, this gives another derivation of (3). We also get a complete classification of quadratic forms over $\mathbb{C}((t))$: a nondegenerate form q is isometric to $\langle 1, \ldots, 1, d(q) \rangle$.

Exercise: Show that as a ring $W(\mathbb{C}((t))) \cong \mathbb{Z}/2\mathbb{Z}[\epsilon]/(\epsilon^2)$. In particular there is exactly one nonzero anisotropic form q such that $q \otimes q$ is hyperbolic: which one?

1.2. $\mathbb{R}((t))$.

Let $\mathbb{R}$ be a field of characteristic different from 2 which is Euclidean, i.e., formally real and with $[\mathbb{R}^\times : \mathbb{R}^{\times 2}] = 2$. We will classify quadratic forms over the field $\mathbb{R}((t))$.

By Lemma X.X, $\mathbb{R}((t))$ has four square classes, represented by $1, -1, t, -t$.

By Springer's Theorem and Sylvester's Law of Inertia (cf. §II.2) we have a group isomorphism

$$W(\mathbb{R}((t)) \cong W(\mathbb{R}) \times W(\mathbb{R}) \cong \mathbb{Z} \times \mathbb{Z}. \tag{4}$$

Let us relate this structural information to the orders on $\mathbb{R}((t))$, using the theory of §II.4. By Theorem II.35, since $\mathbb{R}((t))$ is formally real with torsionfree Witt ring, it is Pythagorean: every sum of squares is already a square. In fact this is easy to see directly: an element $f \in \mathbb{R}((t))^\bullet$ is a square iff

$$f = \sum_{n \geq d} a_n t^n$$

with d even and $a_d > 0$, and a sum of Laurent series of this type is indeed another series of that type. Either way, we get that the total signature map

$$\Sigma : W(\mathbb{R}((t))) \to \prod_{P \in X(\mathbb{R}((t)))} \mathbb{Z}$$

is an injective group homomorphism. Thus $\mathbb{R}((t))$ has at most two orderings.

In fact $\mathbb{R}((t))$ has exactly two orderings: one of them, say P_1, makes a nonzero $f = \sum_{n \geq d} a_n t_n$ is positive iff its lowest degree term a_d is positive (in $\mathbb{R}$, with respect to the unique ordering). In other words, P_1 makes the uniformizer t positive and infinitesimal. On the other hand, $\tau : t \mapsto -t$ induces a field automorphism of $\mathbb{R}((t))$ – the fixed field is $\mathbb{R}((t^2))$. In general, the image of an ordering under a field automorphism is also an ordering, sometimes the ordering we started with and sometimes not. In this case clearly not: we get an ordering P_2 in which the uniformizer $-t$ is positive and infinitesimal. (That was easy. What is less obvious is that there are no more orderings, but as mentioned this follows from Pfister's exact sequence.)

Thus the total signature map may be identified with an injective homomorphism

$$\Sigma = (\sigma_1, \sigma_2) = (\sigma_{P_1}, \sigma_{P_2}) : \mathbb{Z}^2 \hookrightarrow \mathbb{Z}^2.$$

The map Σ is not surjective. In fact, whenever there is more than one ordering, the total signature map cannot be surjective: since our forms are nondegenerate, for any ordering P we have $\sigma_P(q) \equiv \dim q \pmod 2$ and thus for any two orderings P_1 and P_2 on a field we have $\sigma_{P_1}(q) \equiv \sigma_{P_2}(q) \pmod 2$.

Since

$$\Sigma(\langle 1, t\rangle) = (2, 0), \ \Sigma(\langle t\rangle) = (1, -1),$$

we find that $\Sigma(W(\mathbb{R}((t)))$ is the index 2 subgroup $\{(x, y) \in \mathbb{Z}^2 \mid x \equiv y \pmod 2\}$.

Any (nondegenerate, as ever) quadratic form q over $\mathbb{R}((t))$ is isometric to

$$\bigoplus_{i=1}^{a}\langle 1\rangle \oplus \bigoplus_{j=1}^{b}\langle -1\rangle \oplus \bigoplus_{k=1}^{c}\langle t\rangle \oplus \bigoplus_{\ell=1}^{d}\langle -t\rangle \tag{5}$$

for $a, b, c, d \in \mathbb{N}$. Thus we represent quadratic forms over $\mathbb{R}((t))$ be quadruples $(a, b, c, d) \in \mathbb{N}^4$. From our discussion we have that $q_1 \cong q_2$ iff they have the same dimension and total signature, i.e., iff

$$a_1 + b_1 + c_1 + d_1 = a_2 + b_2 + c_2 + d_2,$$
$$a_1 - b_1 + c_1 - d_1 = a_2 - b_2 + c_2 - d_2,$$
$$a_1 - b_1 - c_1 + d_1 = a_2 - b_2 - c_2 + d_2.$$

Exercise: Notice that $\langle t, -t\rangle \cong \langle 1, -1\rangle$, so that in (5) the a, b, c, d are *not* uniquely determined by the isometry class of q. Show that the semigroup $M(\mathbb{R}((t)))$ of nondegenerate quadratic forms is isomorphic to the quotient of $\mathbb{N}^4$ by $(1, 1, 0, 0) \sim (0, 0, 1, 1)$.

We now give a complete analysis of quadratic forms over $\mathbb{R}((t))$.

Because $\mathbb{R}((t))$ is Pythagorean, the elements represented by any form $\langle a_1, \ldots, a_i, c, c, b_1, \ldots, b_j\rangle$ are the same as those of $\langle a_1, \ldots, a_i, c, b_1, \ldots, b_j\rangle$. Thus as far as the representation problem is concerned, we need not consider repeated coefficients, which reduces us to the 16 quadruples (a, b, c, d) with all elements in $\{0, 1\}$. Moreover $q_{(0,0,0,0)}$ is the trivial form; one down. If $a = b = 1$ we have the hyperbolic subform $\langle 1, -1\rangle$, so $q_{(a,b,c,d)}$ is isotropic; similarly, if $c = d = 1$ we have the hyperbolic subform $\langle t, -t\rangle$, so $q_{(a,b,c,d)}$ is isotropic. If $a+b+c+d \geq 3$ then we must have $a+b = 1$ or $c+d = 1$.

Exercise: Show that for each $n \in \mathbb{Z}^+$, there are $(n + 1)^2$ isometry classes of n-dimensional quadratic forms over $\mathbb{R}((t))$.

The form $q_{(1,0,1,0)}$ is P_1-definite hence anisotropic. It represents 1 and t. It does not represent -1 since by the First Representation Theorem $q_{(2,0,1,0)}$ would be isotropic, but it is P_1-definite. Similarly it is not represent $-t$. Thus

$$D(q_{(1,0,1,0)}) = \{1, t\}.$$

Similar arguments show

$$D(q_{(1,0,0,1)}) = \{1, -t\},$$

$$D(q_{(0,1,1,0)}) = \{-1, t\},$$
$$D(q_{(0,1,0,1)}) = \{-1, -t\}.$$

We deduce:
• Every universal form over $\mathbb{R}((t))$ is isotropic. (This is a general property of formally real Pythagorean fields: if $q = \langle a_1, \ldots, a_n \rangle$ represents $-a_1$, then $q \oplus \langle a_1 \rangle$ is isotropic, and then so is q.)

In fact:

• Every anisotropic form over $\mathbb{R}((t))$ represents at most two square classes and only the square classes appearing as the coefficients of any diagonal representation. In particular, any two square classes but $\{1, -1\}$ and $\{t, -t\}$ can be represented.

• Every anisotropic form is (either positive or negative) definite for an ordering (either P_1 or P_2) of $\mathbb{R}((t))$.

• There are four isometry classes of one-dimensional forms, all anisotropic, and each representing a (different) unique square class.

• There are nine isometry classes of two-dimensional forms: the hyperbolic plane; four anisotropic forms each representing a single square class $q_{(2,0,0,0)}$, $q_{(0,2,0,0)}$, $q_{(0,0,2,0)}$, $q_{(0,0,0,2)}$; and four anisotropic forms $q_{(1,0,1,0)}$, $q_{(1,0,0,1)}$, $q_{(0,1,1,0)}$ and $q_{(0,1,0,1)}$ each representing two square classes.

• There are 16 isometry classes of three-dimensional forms. Four of them are isotropic (over any field K, the isotropic three-dimensional forms are all of the form $\mathbb{H} \oplus \langle a \rangle$ and have discriminant $-a$, so correspond bijectively to square classes in K). The anisotropic ones are

$$q_{(3,0,0,0)},\ q_{(0,0,3,0)},\ q_{(0,0,3,0)},\ q_{(0,0,0,3)},$$
$$q_{(2,0,1,0)},\ q_{(1,0,2,0)},$$
$$q_{(2,0,0,1)},\ q_{(1,0,0,2)},$$
$$q_{(0,2,1,0)},\ q_{(0,1,2,0)},$$
$$q_{(0,2,0,1)},\ q_{(0,1,0,2)}.$$

The paired up forms represent the same pair of square classes but are distinguished by their discriminant. In particular, three-dimensional forms over $\mathbb{R}((t))$ are classified by $D(q)$ and $\operatorname{disc} q$. (This is also a general property of formaly real Pythagoren fields.)

There are four similarity classes of 3-dimensional forms; we can pick a representative from each similarity class by requiring the discriminant to be a square, getting the anisotropic ones

$$q_{(3,0,0,0)},\ q_{(1,0,2,0)},\ q_{(1,0,0,2)}$$

and the isotropic one

$$q_{(0,1,1,1)}.$$

We conclude that there are four conics over $\mathbb{R}((t))$; the first three correspond, in order, to the three division quaternion algebras

$$D_1 = \left(\frac{-1,-1}{\mathbb{R}((t))}\right),\ D_2 = \left(\frac{-t,-t}{\mathbb{R}((t))}\right),\ D_3 = \left(\frac{t,t}{\mathbb{R}((t))}\right)$$

over $\mathbb{R}((t))$. In fact

$$\operatorname{Br}\mathbb{R}((t)) \cong \operatorname{Br}\mathbb{R} \oplus \operatorname{Hom}(\mathfrak{g}_{\mathbb{R}}, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z},$$

so every noncommutative $\mathbb{R}((t))$-central division algebra is a quaternion algebra.

Make a table of Hilbert symbols

Exercise:
a) Show that the field extension $\mathbb{C}((t))/\mathbb{R}((t))$ splits every quaternion algebra.
b) Show that in fact there are no division quaternion algebras over $\mathbb{C}((t))$. Indeed, show $\operatorname{Br}\mathbb{C}((t)) = 0$.
c) Show that the quaternion algebra D_1 is not split by either $\mathbb{R}((\sqrt{t}))$ or $\mathbb{R}(\sqrt{-t}))$, so has exactly one quadratic splitting field.
d) Show that the quaternion algebra D_2 is not split by $\mathbb{R}((\sqrt{t}))$ and is split by $\mathbb{R}((\sqrt{-t}))$, so has exactly two quadratic splitting fields.
e) Show that the quaternion algebra D_3 is split by $\mathbb{R}((\sqrt{t}))$ and is not split by $\mathbb{R}((\sqrt{-t}))$, so has exactly two quadratic splitting fields.

2. Murderizing quadratic forms over non-dyadic local fields

Throughout this section we specialize to the case in which R is a nondyadic Henselian DVR with *finite* residue field $\mathbb{F}_q$. (Note that the nondyadic hypothesis is equivalent to q being odd.) In this case the results of the previous section give an extremely explicit description of all quadratic forms over K, and this description is extremely useful. Otherwise put, using what we now know we can **murderize** quadratic forms over K – so, in particular, over $\mathbb{Q}_p$ for odd p – and we aim to do so!

By Theorem 6, anisotropic quadratic forms q over K correspond to pairs of anisotropic quadratic forms over $k = \mathbb{F}_q$. Since we know there are exactly four anisotropic quadratic forms over $\mathbb{F}_q$ – including the zero-dimensional form, as always! – it follows that there are $4^2 = 16$ anisotropic quadratic forms over K.

Let $r \in \mathbb{F}_q^\times \setminus \mathbb{F}_q^{\times 2}$. If (and only if) $q \equiv 3 \pmod 4$, we may choose $r = -1$; let us agree to do so in that case.

The four anisotropic quadratic forms over $\mathbb{F}_q$ are:

- The zero form 0.
- The two one-dimensional forms x^2 and rx^2.
- The two-dimensional form $x^2 - ry^2$.

Of course $x^2 - ry^2$ is anisotropic since r is not a square in $\mathbb{F}_q$. But here is another way to look at it: a binary form is isotropic iff it is hyperbolic iff it has discriminant -1. Our form has discriminant $-r$, which, since r is not a square, is not in the same square class as -1.

The square classes in K are represented by $1, r, \pi, r\pi$.

Now let us write down all the anisotropic forms over K and what square classes they represent!

0.1: The zero form. (It doesn't represent anything.)
1.1: The form x^2. It represents the square class 1.
1.2: The form rx^2. It represents the square class r.
1.3: The form πx^2. It represents the square class p.
1.4 The form $r\pi x^2$. It represents the square class $r\pi$.

On to the binary forms. We can be even more murderous than simply writing down representatives for the 6 anisotropic binary forms. In fact we can – and might as well! – write down all 10 different-looking binary forms and determine all isomorphisms between them, an/isotropy, and all square classes represented. The ten forms in question are:

2.1 $\langle 1, 1\rangle$.
2.2 $\langle 1, r\rangle$.
2.3 $\langle 1, \pi\rangle$.
2.4 $\langle 1, r\pi\rangle$.
2.5 $\langle r, r\rangle$.
2.6 $\langle r, \pi\rangle$.
2.7 $\langle r, r\pi\rangle$.
2.8 $\langle \pi, \pi\rangle$.
2.9 $\langle \pi, r\pi\rangle$.
2.10 $\langle r\pi, r\pi\rangle$.

We claim that any anisotropic such form represents precisely two of the four square classes in K. Indeed, consider $\langle a, b\rangle$.
Case i: If $a, b \in R^\times$, then by our description of $W(K)$, $ax^2 + by^2 + u\pi z^2 = 0$ are anisotropic for $u \in R^\times$, so $\langle a, b\rangle$ does not represent $\pi, r\pi$. Similarly $ax^2+by^2+uz^2 = 0$ is isotropic, so $\langle a, b\rangle$ represents $1, r$.
Case 2: if $a \in R^\times$ and $b = \pi u$, $u \in R^\times$, then for $v \in R^\times$ $ax^2 + u\pi y^2 + vz^2 = 0$ is isotropic iff $-av \in k^{\times 2}$, so it represents one out of the two unit square classes. Moreover $ax^2 + u\pi y^2 + v\pi z^2 = 0$ is isotropic iff $-uv \in k^{\times 2}$, so it represents one out of the two non-unit square classes.
Case 3: If $a = \pi u$, $b = \pi v$, then $\langle a, b\rangle = \pi\langle u, v\rangle$, so by Case 1 it represents precisely the two nonunit square classes.

Now, the isotropy of some of these forms depends upon whether $q \equiv \pm 1 \pmod 4$, so for proper murderization we consider these cases separately (serially?).

Case 1: $q \equiv 1 \pmod 4$. Then:

2.1 has discriminant $1 \equiv -1 \pmod{k^\times}$, so is isotropic (and thus represents all four square classes).

2.2 has discriminant $r \equiv -r$ so is anisotropic. It represents the square classes $\{1, r\}$.
2.3 has discriminant π so is anisotropic. It represents the square classes $\{1, \pi\}$.
2.4 has discriminant $r\pi$ so is anisotropic. It represents the square classes $\{1, r\pi\}$.
2.5 has discriminant $1 \equiv -1$ so is isotropic.
2.6 has discriminant $r\pi$ so is anisotropic. It represents the square classes $\{1, r\pi\}$.
2.7 has discriminant π so is anisotropic. It represents the square classes $\{r, r\pi\}$.
2.8 has discriminant 1 so is isotropic.
2.9 has discriminant r so is anisotropic. It represents the square classes $\{\pi, r\pi\}$.
2.10 has discriminant 1 so is isotropic.

Let us retally the anisotropic square classes in the $q \equiv 1 \pmod 4$ case:

2_1.1: $\langle 1, r\rangle$ represents $\{1, r\}$.
2_1.2: $\langle 1, \pi\rangle$ represents $\{1, \pi\}$.
2_1.3: $\langle 1, r\pi\rangle$ represents $\{1, r\pi\}$.
2_1.4: $\langle r, \pi\rangle$ represents $\{r, \pi\}$.
2_1.5: $\langle r, r\pi\rangle$ represents $\{r, r\pi\}$.
2_1.6: $\langle \pi, r\pi\rangle$ represents $\{\pi, r\pi\}$.

Case 2: $q \equiv 3 \pmod 4$. Recall that we take $r = -1$ here. Then:

2.1 has discriminant $1 \not\equiv -1 \pmod{k^\times}$, so is anisotropic. It represents the square classes $\{1, -1\}$.
2.2 has discriminant $r = -1$ so is isotropic.
2.3 has discriminant π so is anisotropic. It represents the square classes $\{1, \pi\}$.
2.4 has discriminant $-\pi$ so is anisotropic. It represents the square classes $\{1, -\pi\}$.
2.5 has discriminant 1 so is anisotropic. It represents the square classes $\{1, -1\}$.
2.6 has discriminant $-\pi$ so is anisotropic. It represents the square classes $\{1, -\pi\}$.
2.7 has discriminant π so is anisotropic. It represents the square classes $\{-1, -\pi\}$.
2.8 has discriminant 1 so is anisotropic. It represents the square classes $\{\pi, -\pi\}$.
2.9 has discriminant -1 so is isotropic.
2.10 has discriminant 1 so is anisotropic. It represents the square classes $\{\pi, -\pi\}$.

But this gives us eight anisotropic forms: two too many! Two of them must be isomorphic, and the only possible pairs are the one which represent the same square classes. Indeed, both 2.1 and 2.5 have the same discriminant and represent a common value so are isomorphic, and the same goes for 2.8 and 2.10.

We retally so as to list only distinct anisotropic forms when $q \equiv 3 \pmod 4$:

2_3.1: $\langle 1, 1\rangle \cong \langle -1, -1\rangle$ represents $\{1, -1\}$.
2_3.2: $\langle 1, \pi\rangle$ represents $\{1, \pi\}$.
2_3.3: $\langle 1, -\pi\rangle$ represents $\{1, -\pi\}$.
2_3.4: $\langle -1, \pi\rangle$ represents $\{-1, \pi\}$.
2_3.5: $\langle -1, -\pi\rangle$ represents $\{-1, -\pi\}$.
2_3.6: $\langle \pi, \pi\rangle \cong \langle -\pi, -\pi\rangle$ represents $\{\pi, -\pi\}$.

Notice that in each of the two cases we got, as advertised, precisely six classes of anisotropic binary forms. Morever, we worked out above that any anisotropic binary form represents precisely two out of the four square classes of K, and in fact even more is true: of the $6 = \binom{4}{2}$ 2-element subsets of $\{1, r, \pi, r\pi\}$, each of them is the set of square classes represented by a unique anisotropic binary form!

Ternary forms: Because we murderized the binary forms, understanding the ternary forms is easy. To get an anisotropic ternary form we must start with an anisotropic binary form $\langle a, b\rangle$ and add on c such that $\langle a, b\rangle$ does not represent the square class $-c$. Thus each of the 6 anisotropic binary forms can be escalated to anisotropic ternary forms in two different ways, giving 12 ternary forms in all. It happens that there are only four distinct isomorphism classes here, so the 12 forms "come together" in groups of 3. Further, each of these four isomorphism classes of anisotropic ternary forms represents exactly three out of the four square classes.

Again, we treat $q \equiv 1 \pmod 4$ and $q \equiv 3 \pmod 4$ separately.

Case 1: $q \equiv 1 \pmod 4$:

For instance, the form $2_1.1$: $\langle 1, r\rangle$ can be escalated to a ternary form by adding on $-\pi \equiv \pi$ and $-r\pi \equiv r\pi$, giving us the two anisotropic forms:

$3_1.1.1$: $\langle 1, r, \pi\rangle$
$3_1.1.2$: $\langle 1, r, r\pi\rangle$.

Doing the same with the other six forms gives:

$3_1.2.1$: $\langle 1, \pi, r\rangle$,
$3_1.2.2$: $\langle 1, \pi, r\pi\rangle$,
$3_1.3.1$: $\langle 1, r\pi, r\rangle$,
$3_1.3.2$: $\langle 1, r\pi, \pi\rangle$,
$3_1.4.1$: $\langle r, \pi, 1\rangle$,
$3_1.4.2$: $\langle r, \pi, r\pi\rangle$,
$3_1.5.1$: $\langle r, r\pi, 1\rangle$,
$3_1.5.2$: $\langle r, r\pi, \pi\rangle$,
$3_1.6.1$: $\langle \pi, r\pi, 1\rangle$,
$3_1.6.2$: $\langle \pi, r\pi, r\rangle$.

Looking at these 12 forms, it is now clear how they "triple up": they all have distinct entries, so four each square class s, the three forms which omit s but contain the other three square classes are obviously equivalent!

Note that the fact that these ternary anisotropic forms have coefficients lying in distinct square classes is a consequence of -1 being a square in K: because of this, having repeated coefficients is equivalent to having coefficients $x, -x$ which gives a hyperbolic plane inside q.

Therefore the most reasonable way to index these forms seems to be by the omitted coefficient. Moreover, recall that no anisotropic ternary form $q = \langle a, b, c\rangle$ represents the square class $-abc = -d(q)$. Since, again, -1 is a square, each of these anisotropic ternary forms fails to represent a unique square class: the one which does not appear as a diagonal coefficient. Thus:

$3_1.(1)$: $\langle r, \pi, r\pi\rangle$ represents all but 1.
$3_1.(r)$: $\langle 1, \pi, r\pi\rangle$ represents all but r.
$3_1.(\pi)$: $\langle 1, r, r\pi\rangle$ represents all but π.
$3_1.(r\pi)$: $\langle 1, r, \pi\rangle$ represents all but $r\pi$.

Now we turn to the case $q \equiv 3 \pmod 4$. Performing the same escalation process gives us 6 pairs of anisotropic ternary forms:

$3_3.1.1$: $\langle 1, 1, \pi\rangle$
$3_3.1.2$: $\langle 1, 1, -\pi\rangle$
$3_3.2.1$: $\langle 1, 1, \pi\rangle$
$3_3.2.2$: $\langle 1, \pi, \pi\rangle$
$3_3.3.1$: $\langle 1, 1, -\pi\rangle$
$3_3.3.2$: $\langle 1, -\pi, -\pi\rangle$
$3_3.4.1$: $\langle -1, -1, \pi\rangle$
$3_3.4.2$: $\langle -1, \pi, \pi\rangle$
$3_3.5.1$: $\langle -1, -1, -\pi\rangle$
$3_3.5.2$: $\langle -1, -\pi, -\pi\rangle$
$3_3.6.1$: $\langle -1, \pi, \pi\rangle$
$3_3.6.2$: $\langle 1, \pi, \pi\rangle$

Now we perform the tripling up process.

Discriminant 1: the forms $3_3.2.2$ and $3_3.6.2$ are both $\langle 1, \pi, \pi\rangle$. The other form of discriminant 1 is $3_3.3.2$: $\langle 1, -\pi, -\pi\rangle$. But according to our list of anisotropic binary forms, $\langle \pi, \pi\rangle \cong \langle -\pi, -\pi\rangle$, so $\langle 1, \pi, \pi\rangle \cong \langle 1, -\pi, -\pi\rangle$. It is now also clear that this form represents the square classes $1, \pi, -\pi$ and, like any anisotropic ternary form, does not represent $-\operatorname{disc} q = -1$.

Discriminant -1: the forms $3_3.4.2 = 3_3.6.1$ are both $\langle -1, \pi, \pi\rangle$. The other form of discriminant -1 is $3_3.5.2$: $\langle -1, -\pi, -\pi\rangle$, and as above since $\langle \pi, \pi\rangle \cong \langle -\pi, -\pi\rangle$, it is clear that these forms are equivalent, represent the three square classes $-1, \pi, -\pi$ and do not represent 1.

Discriminant π: the forms $3_3.1.1 = 3_3.2.1$ are both $\langle 1, 1, \pi$. The other form of discriminant π is $3_3.4.1$: $\langle -1, -1, \pi\rangle$. Since $\langle 1, 1\rangle \cong \langle -1, -1\rangle$, these forms are equivalent, represent the three square classes $1, -1, \pi$, and do not represent $-\pi$.

Discriminant $-\pi$: the forms $3_3.3.1 = 3_3.5.1$ are both $\langle 1, 1, -\pi\rangle$. The other form of discriminant $-\pi$ is $3_3.5.1$: $\langle -1, -1, -\pi\rangle$. Since $\langle 1, 1\rangle \equiv \langle -1, -1\rangle$, these forms are equivalent, represent the three square classes $1, -1, -\pi$, and do not represent π.

Thus:

$3_3.(1)$: $\langle 1, \pi, \pi\rangle \cong \langle 1, -\pi, -\pi\rangle$ represents all but -1.

$3_3.(-1)$: $\langle -1, \pi, \pi \rangle \cong \langle -1, -\pi, -\pi \rangle$ represents all but 1.
$3_3.(\pi)$: $\langle 1, 1, \pi \rangle \cong \langle -1, -1, \pi \rangle$ represents all but $-\pi$.
$3_3.(-\pi)$: $\langle 1, l, -\pi \rangle \cong \langle -1, -\pi, -\pi \rangle$ represents all but π.

Quaternary forms:

Every anisotropic quaternary form is obtained by passing from an anisotropic ternary form $q = \langle a, b, c \rangle$ to $\langle a, b, c, d \rangle$, where q does not represent $-d$.

Case $q \equiv 1 \pmod 4$: since -1 is a square, the unique way of completing each anisotropic ternary form to an anistropic quaternary form is by taking d to be $-\operatorname{disc}(q) = \operatorname{disc} q$. In each case we get a quaternary form whose coefficients are the distinct square classes, so the unique anisotropic quaternary form is

$4_1.1$: $\langle 1, r, \pi, r\pi \rangle$.

This form is universal.

Case $q \equiv 3 \pmod 4$: performing the same escalation process with each of our four anisotropic ternary forms, we get four superficially different anisotropic ternary forms:

$\langle 1, 1, \pi, \pi \rangle$
$\langle 1, 1, -\pi, -\pi \rangle$
$\langle -1, -1, \pi, \pi \rangle$
$\langle -1, -1, -\pi, -\pi \rangle$.

Since $\langle 1, 1 \rangle \equiv \langle -1, -1 \rangle$ and $\langle \pi, \pi \rangle \equiv \langle -\pi, -\pi \rangle$, all four forms above are equivalent, so up to isomorphism there is again a unique anisotropic quaternary form:

$4_3.1$: $\langle 1, 1, \pi, \pi \rangle = \langle 1, 1, -\pi, -\pi \rangle = \langle -1, -1, \pi, \pi \rangle = \langle -1, -1, -\pi, -\pi \rangle$.

This form is universal.

The murderization is now complete.

3. The Local Square Theorem

Let K be a Henselian discretely valued field, with valuation ring R, (choice of a) uniformizing element π, and residue field k. For $x \in K^\times$, put $|x| = e^{-v(x)}$, so $|\cdot|$ is a non-Archimedean absolute value.

We recall the following simple form of Hensel's Lemma (see e.g. XX).

Proposition 9. *(Hensel's Lemma) Let $f \in R[t]$ be a polynomial. Suppose there is $\alpha \in R$ such that $v(f(\alpha)) > 2v(f'(\alpha))$. Then there is $\beta \in R$ such that $f(\beta) = 0$.*

Theorem 10. *(Local Square Theorem) Let K be a Henselian discretely valued field, with valuation ring R and uniformizer π. For any $\alpha \in R$, $1 + 4\pi\alpha \in R^{\times 2}$.*

Proof. [G, Thm. 3.39] Let $f(t) = \pi t^2 + t - \alpha$. Then

$$v(f(\alpha)) = v(\pi\alpha^2) = 1 + 2v(\alpha) \geq 1 > 0 - v(2\pi\alpha + 1) = v(f'(\alpha)),$$

so by Hensel's Lemma there exists $\beta \in R$ such that $f(\beta) = 0$. Therefore the discriminant $1 + 4\pi\alpha$ of f is a square in K, so $1 + 4\pi\alpha \in K^{\times 2} \cap R = R^{\bullet 2}$. □

3.1. **Orderings on Laurent Series Fields.**

The Local Square Theorem can be applied to recover and generalize some of the results of §X.X.

Proposition 11. [L, Prop. VIII.4.11] *Let k be a field.*
a) Each ordering P on k extends to exactly two orderings on the Laurent series field $k((t))$: an ordering P_1 in which t is positive and smaller than every $x \in P$ and an ordering P_2 in which $-t$ is positive and smaller than every $x \in P$.
b) The field k is formally real Pythagorean iff $k((t))$ is.
c) If $\mathbb{R}$ is Euclidean, then the iterated Laurent series field $\mathcal{L}_n = \mathbb{R}((t_1))\cdots((t_n))$ has 2^{n+1} square classes, 2^n orderings and has $W(\mathcal{L}_n) \cong \mathbb{Z}^{2^n}$.

Proof. We will prove part a) only. Parts b) and c) are similar enough to what has been done for $\mathbb{R}((t))$ so as to make good exercises.

Let

$$P_1 = \{f \in k((t))^\bullet \mid f = \sum_{n\geq N} a_n t^n \text{ with } a >_P 0\}.$$

It is immediate that P_1 satisfies the properties (PC1) through (PC4) and thus is (the positive cone associated to) an ordering on $k((t))$. Certainly $t \in P_1$. Conversely, let $\mathcal{P}$ be an ordering on $k((t))$ containing t and such that $\mathcal{P} \cap k = P$. By the Local Square Theorem, $1 + \sum_{n\geq 1} a_n t^n$ is a square in $k[[t]]$ hence also in $k((t))$. Then if $a_N \in P$ we have

$$\sum_{n\geq N} a_n t^n = a_N t^N \left(1 + \sum_{n\geq 1} \frac{a_{N+n}}{a_N} t^N\right) \in \mathcal{P} \iff a_N \in P.$$

The map $\tau : k((t)) \to k((t))$ by $\sum_{n\geq N} a_n t^n \mapsto \sum_{n\geq N} a_n (-t)^n$ is an order two field automorphism leaving k pointwise fixed. Thus $P_2 = \tau(P_1)$ is an ordering of $k((-t)) = k((t))$ containing $P \cup \{-t\}$. □

Exercise: Complete the proof of Proposition 11.

4. The Hilbert Symbol, The Hilbert Invariant and Applications

In this section K is a field endowed with a norm $|\cdot|$ with respect to which it is locally compact and not discrete. That is, K is either the real or complex numbers, a finite extension of $\mathbb{Q}_p$, or $\mathbb{F}_q((t))$. (The first two are trivial cases, and the reasons for their inclusion here will become clear only later when we discuss global fields.)

4.1. **The Hilbert Symbol.**

Let $a, b \in K$. We define the **Hilbert symbol** (a, b) to be 1 if the quadratic form $ax^2 + by^2$ represents 1 and -1 otherwise. Equivalently, we define it to be 1 (resp. -1) if the ternary form $ax^2 + by^2 - z^2$ is isotropc (resp. anisotropic).

Proposition 12. *(First Properties of Hilbert Symbols) Let* $a, b, c, d \in K^\times$*.*
a) If $a \equiv c \pmod{K^{\times 2}}$ *and* $b \equiv d \pmod{K^{\times 2}}$*, then* $(a, b) = (c, d)$*. In other words, the Hilbert symbol factors through* $K^\times/K^{\times 2} \times K^\times/K^{\times 2}$*.*
b) $(a, b) = (b, a)$*.*
c) $(a^2, b) = 1$*.*
d) $(a, -a) = (a, 1 - a) = 1$*.*

Exercise: Prove Proposition 12.

Exercise (Hilbert symbols over $\mathbb{R}$ and $\mathbb{C}$):
a) Let $K = \mathbb{C}$. Show that for all $a, b \in \mathbb{C}^\times$, $(a, b) = 1$.
b) Let K be formally real. Show that if $(a, b) = 1$, then a and b are not both negative.
c) Let K be real-closed (e.g. $K = \mathbb{R}$!). Show that if a and b are not both negative, $(a, b) = 1$.

Exercise (non-dyadic Hilbert symbols): a) Let K be a non-dyadic local field with residue field $\mathbb{F}_q$. Make a 4×4 table giving the values of the Hilbert symbol (a, b) as a and b each run over all square classes $\{1, r, \pi, \pi r\}$. (Some of the entries in your table well depend upon whether q is 1 or -1 modulo 4.)
b) Choose a uniformizing element π. Using this choice, for $a \in K^\times$, put $u_a = \frac{a}{\pi^{v(a)}}$. Show that for $a, b \in K^\times$,

$$(a, b) = (-1)^{v(a)v(b)\frac{q-1}{2}} \left(\frac{u_a}{q}\right)^{v(b)} \left(\frac{u_b}{q}\right)^{v(a)}.$$

d) Viewing the Hilbert symbol as a map $K^\times/K^{\times 2} \times K^\times/K^{\times 2} \to \{\pm 1\}$, show it is :
(i) bilinear: $(xy, z) = (x, z)(y, z)$, $(x, yz) = (x, z)(y, z)$ and
(ii) nondegenerate: if $(x, y) = 1$ for all $y \in K^\times$, then $x \in K^{\times 2}$.

Exercise (Hilbert symbols in $\mathbb{Q}_2$): Let $K = \mathbb{Q}_2$.
a) Fill in the 8×8 table of (a, b) as a and b each run over all square classes $\{1, 2, 3, 5, 6, 7, 10, 14\}$ of $\mathbb{Q}_2$.
b) Show that...
c) Show that the Hilbert symbol is a nondegenerate bilinear form.

The previous two exercises show, in particular, that the Hilbert symbol: $K^\times \times K^\times \to \{\pm 1\}$ is a *Steinberg symbol* in the sense of X.X, when K is either a nondyadic locally compact field or $\mathbb{Q}_2$. Accordingly, we may define a **Hilbert invariant**, in a manner we will review in the next section.

Now let K be a proper, finite extension of $\mathbb{Q}_2$. For $a, b \in K^\times$ we define (a, b) exactly as above: namely as $+1$ if the form $ax^2 + by^2 - z^2$ is isotropic and -1 if it is anisotropic. It turns out that again this gives a Steinberg symbol, but to show this requires more than the very elementary calculations done above. Further, the Hilbert symbol is nondegenerate as a bilinear map $K^\times/K^{\times 2} \times K^\times/K^{\times 2} \to \{\pm 1\}$. If we assume these facts for now, then we will be able to give a complete classification of quadratic forms over locally compact fields that includes the general dyadic case. Later we will go back and explain how these facts about the Hilbert symbol follow from standard – but deep – facts of the arithmetic of local fields.

4.2. **The Hilbert Invariant.**

Let $q = \langle a_1, \ldots, a_n \rangle$ be a regular quadratic form over the non-discrete, locally compact field K. We define the **Hilbert invariant**

$$H(a_1, \ldots, a_n) = \prod_{i<j} (a_i, a_j).$$

(When $n = 1$, we set $H(a_1) = 1$.) Our first order of business is to show that H is indeed an invariant, i.e., that it depends only on the isometry class of q and not the chosen diagonalization. For this we need a preliminary result which will be useful in its own right.

Lemma 13. *Let $q = \langle a_1, a_2 \rangle$ be a regular binary form, and let $b \in K^\times$. TFAE:*
(i) q represents b.
(ii) $(b, -\operatorname{disc} q) = (a_1, a_2)$.

Proof. q represents b iff $a_1x^2 + a_2y^2 - bz^2 = 0$ is isotropic iff $a_1bx^2 + a_2by^2 - z^2 = 0$ is isotropic iff $1 = (a_1b, a_2b) = (a_1, a_2)(a_1, b)(a_2, b)(b, b) = (a_1, a_2)(\operatorname{disc} q, b)(-1, b) = (a_1, a_2)(-\operatorname{disc} q, b)$. □

Proposition 14. *Let $\langle a_1, \ldots, a_n, b_1, \ldots, b_n \in K^\times$. If $\langle a_1, \ldots, a_n \rangle \cong \langle b_1, \ldots, b_n \rangle$, then $H(a_1, \ldots, a_n) = H(b_1, \ldots, b_n)$.*

Proof. The result is trivial for $n = 1$.
Step 1: Suppose $n = 2$, and $q = \langle a_1, a_2 \rangle \cong \langle b_1, b_2 \rangle$. Then q represents b_1, so by Lemma 13 $(a_1, a_2) = (b_1, -\operatorname{disc} q) = (b_1, -b_1b_2) = (b_1, -b_1)(b_1, b_2) = (b_1, b_2)$.
Step 2: Suppose $n > 2$. By the Chain Equivalence Theorem, we may suppose that $a_i \neq b_i$ for at most two values of i. Further, since $\prod_{i<j}(a_i, a_j)$ is independent of the ordering of $a_1, \ldots, a_n$, we may suppose $a_i = b_i$ for all $i > 2$ and (by Witt Cancellation) that $\langle a_1, a_2 \rangle \cong \langle b_1, b_2 \rangle$. Thus $a_1a_2 \equiv b_1b_2 \pmod{K^{\times 2}}$ and $(a_1, a_2) = (b_1, b_2)$ by Step 1. Thus

$$\prod_{i<j}(a_i, a_j) = (a_1, a_2) \prod_{j>2}(a_1a_2, a_j) \prod_{2<i<j}(a_i, a_j)$$
$$= (b_1, b_2) \prod_{j>2}(b_1b_2, b_j) \prod_{2<i<j}(b_i, b_j) = \prod_{i<j}(b_i, b_j).$$

□

In view of Proposition 14 we may write $H(q)$ instead of $H(a_1, \ldots, a_n)$, and we call it the **Hilbert invariant** of q. As we are about to see, the Hilbert invariant is the key piece of information beyond the dimension and the discriminant needed to classify quadratic forms over Henselian fields with finite residue fields.

Proposition 15. *For forms f, g over K, we have*

$$H(f \oplus g) = (\operatorname{disc} f, \operatorname{disc} g) H(f) H(g).$$

Proof. Writing $f = \langle a_1, \ldots, a_m \rangle$, $g = \langle b_1, \ldots, b_n \rangle$, we have

$$H(f \oplus g) = \prod_{i<j}(a_i, a_j) \prod_{i<j}(b_i, b_j) \prod_{i,j}(a_i, b_j)$$
$$= H(f)H(g)(\prod_i a_i, \prod_i, b_j) = H(f)H(g)(\operatorname{disc} f, \operatorname{disc} g).$$

□

4.3. Applications of the Hilbert Invariant.

Throughout this section K denotes a Henselian discretely valued field with finite residue field $\mathbb{F}_q$.

Lemma 16. *A ternary $q_{/K}$ is isotropic iff $H(q) = (-1, -\operatorname{disc} q)$.*

Proof. Write $q(x, y, z) = ax^2 + by^2 + cz^2$. First note that

$$(-ac, -bc) = (a, b)(a, -c)(b, -c)(-c, -c) = (a, b)(ab, -c)(-c, -1)$$
$$= (a, b)(-ab, -c) = (a, b)(-1, -1)(ab, c)(-1, abc).$$

Thus

$$H(q)(-1, -\operatorname{disc} q) = (a, b)(a, c)(b, c)(-1, -abc)$$
$$= (a, b)(ab, c)(-1, -abc) = (a, b)(-1, -1)(ab, c)(-1, abc) = (-ac, -bc).$$

Since q is isotropic if $-acx^2 - bcy^2 - z^2 = 0$ iff $(-ac, -bc) = 1$, the result follows. □

Lemma 17. *A quaternary $q_{/K}$ is anisotropic iff $\operatorname{disc} q \in K^{\times 2}$ and $H(q) = -(-1, -1)$.*

Proof. We may write $q = g(x) - h(y) = a_1x_1^2 + a_2x_2^2 - b_1y_1^2 - b_2y_2^2$. We claim q is isotropic iff there exists $d \in K^\times$ which is simultaneously represented by g and h. It is immediate that if this holds then q is isotropic. Conversely, if q is isotropic there are $v, w \in K^2$, not both zero, such that $g(v) = h(w)$. If this common value is nonzero, then it is the d we want. If this common value is zero, then one of g and h is the hyperbolic plane, hence universal, and the result is trivial.
Now, by Lemma 13, f and g both represent $d \in K^\times$ iff

$$(d, -a_1a_2) = (a_1, a_2),$$
$$(d, -b_1b_2) = (b_1, b_2).$$

Note that $\langle a_1, a_2\rangle$ is hyperbolic iff $-a_1a_2 \in K^{\times 2}$. In this case, $(a_1, a_2) = (a_1, -a_1) = 1$ and the first equation holds for all d. In this case q contains a hyperbolic subform so is isotropic. Similarly for $\langle b_1, b_2\rangle$. Now assume that $-a_1a_2$ and $-b_1b_2$ are both nonsquares: then $(d, -a_1a_2) = (a_1, a_2)$ and $(d, -b_1b_2) = (b_1, b_2)$ each hold for precisely half of the square classes, and q is anisotropic iff these sets of d's are complementary. We claim this occurs iff $a_1a_2K^{\times 2} = b_1b_2K^{\times 2}$ and $(a_1, a_2) = -(b_1, b_2)$. This is perhaps best seen by viewing $K^\times/K^{\times 2}$ as a finite dimensional $\mathbb{Z}/2\mathbb{Z}$-vector space and the two loci as affine hyperplanes in that space. Two affine hyperplanes do not intersect iff they are distinct and parallel, giving the above conditions.

The condition $a_1a_2K^{\times 2} = b_1b_2K^{\times 2}$ gives $\operatorname{disc} q \in K^{\times 2}$, and the condition $(a_1, a_2) = -(b_1, b_2)$ gives

$$H(f) = -(b_1, b_2)(a_1, -b_1)(a_1, -b_2)(a_2, -b_1)(a_2, -b_2)(-b_1, -b_2)$$
$$= -(b_1, b_2)(a_1, b_1)(a_1, -1)(a_1, b_2)(a_1, -1)(a_2, b_1)(a_2, -1)(a_2, b_2)(a_2, -1)(-1, -1)(-1, b_2)(-1, b_1)(b_1, b_2)$$
$$= -(-1, -1)(a_1, b_1b_2)(a_2, b_1b_2)(-1, b_1b_2)$$
$$= -(-1, -1)(-a_1a_2, b_1b_2) = -(-1, -1)(-a_1a_2, a_1a_2) = -(-1, -1).$$

□

Corollary 18. *A ternary $q_{/K}$ represents all square classes except possibly $-\operatorname{disc} f$.*

Proof. Indeed, if $dK^{\times 2} \neq (-\operatorname{disc} q)K^{\times 2}$, then the ternary form $q(x, y, z) - dw^2$ has nonsquare discriminant so must be isotropic. □

Corollary 19. *A form $q_{/K}$ in at least five variables is isotropic.*

Proof. We may write $q = \langle a_1, a_2, a_3, a_4, a_5 \rangle$ and it suffices to find $d \in K^\times$ which is simultaneously represented by the ternary form $\langle a_1, a_2, a_3 \rangle$ and the binary form -$\langle a_4, a_5 \rangle$. But indeed the ternary form represents all but possibly one square class, and the binary form represents at least half of the square classes, hence at least two square classes, so there must be a square class represented by both. ☐

Theorem 20. *The dimension, discriminant and Hilbert invariant is a complete system of invariants for regular quadratic forms over K. That is, for regular quadratic forms f, g over K, TFAE:*
(i) $f \cong g$.
(ii) $\dim f = \dim g$, $\operatorname{disc} f = \operatorname{disc} g$ *and* $H(f) = H(g)$.

Proof. (i) $\implies$ (ii) is clear.
(ii) $\implies$ (i): The case of $n = 1$ is trivial. Suppose $f = \langle a_1, a_2 \rangle$ and $g = \langle b_1, b_2 \rangle$ have the same discriminant and the same Hilbert invariant. By Lemma 13, $(a_1, a_2) = (b_1, b_2) = (b_1, -\operatorname{disc} g) = (b_1, -\operatorname{disc} f)$, so f represents b_1. Therefore f and g, being binary forms of the same discriminant representing a common value, are isometric. Now suppose $n \geq 3$. Then the form $f(x) - g(y)$ has at least six variables so is isotropic, hence as in the proof of Lemma 17 f and g represent a common value $d \in K^\times$. Therefore we may write $f = \langle d \rangle \oplus f_1$, $g = \langle d \rangle \oplus g_1$. Clearly f_1 and g_1 have the same dimension and the same discriminant, and by Proposition 15 they have the same Hasse invariant. The result now follows by induction on n. ☐

Having shown that the dimension, discriminant and Hilbert invariant serve to classify forms over K, a natural followup question is what values these invariants can take. Clearly in dimension one the Hasse invariant is 1; also, since any binary form of discriminant -1 is isomorphic to $\langle 1, -1 \rangle$, the Hasse invariant of any such form is 1. It turns out that these are the only restrictions.

Theorem 21. *Let q be a quadratic form over K.*
a) If $\dim q = 1$, $H(q) = 1$.
b) If $\dim q = 2$ *and* $\operatorname{disc} q = -1$ *then* $H(q) = 1$. *For any* $d \not\equiv -1 \pmod{K^{\times 2}}$ *and any* $\epsilon \in \{\pm 1\}$, *there is a binary form q with* $\operatorname{disc} q = d$ *and* $H(q) = \epsilon$.
c) For any $n \geq 3$, $d \in K^\times / K^{\times 2}$ *and* $\epsilon \in \{\pm 1\}$, *there is a form q with* $\dim q = n$, $\operatorname{disc} q = d$ *and* $H(q) = \epsilon$.

Proof. a) This is clear.
b) As above, this follows because any binary form of discriminant -1 is isometric to the hyperbolic plane $\langle 1, -1 \rangle$ and thus has trivial Hasse invariant. Now take $d \neq -1$; for any $a \in K^\times$, $f = \langle a, ad \rangle$ has discriminant d and Hilbert invariant $(a, ad) = (a, a)(a, d) = (-1, a)(d, a) = (-d, a)$. Because $-d$ is not a square, we can choose a so as to make the Hilbert symbol either ± 1.
c) Suppose $n \geq 3$ and the result has been shown for all forms of dimension $n - 1$. Fix $d \in K^\times$, and choose $a \in K^\times$ such that $-ad$ is not a square. Consider $\langle a \rangle \oplus g$ with $\operatorname{disc}(g) = ad$. Then $\operatorname{disc} f = d$ and $H(f) = (a, ad) H(g)$. By induction we may choose g such that $H(g)$ has arbitrary sign, and so $H(f)$ can have both signs. ☐

Corollary 22. *Let K be a locally compact, discretely valued field. Let $2^\delta = \# K^\times / K^{\times 2}$.*
a) There is one anisotropic form of dimension zero.

b) There are 2^δ *anisotropic forms of dimension one.*
c) There are $2(2^\delta - 1)$ *anisotropic forms of dimension two.*
d) There are 2^δ *anisotropic forms of dimension three.*
e) There is one anisotropic form of dimension four.
Thus $\#W(K) = 2^{\delta+2}$.

Proof. Exercise. □

Exercise: Suppose K is non-dyadic. Then $\#K^\times/K^{\times 2} = 4$, so $\#W(K) = 16$. Show (again!) that $W(K) \cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$.

Exercise: Suppose $K = \mathbb{Q}_2$. Then $\#K^\times/K^{\times 2} = 8$, so $\#W(K) = 32$. Show that $W(\mathbb{Q}_2) \cong \mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

5. Quadratic forms over $\mathbb{R}((t))$

Theorem 23. *Let k be a field of characteristic not 2.*
a) Every ordering on k extends to exactly two orderings on $k((t))$: one in which $t > 0$ and one in which $t < 0$.
b) The field k is formally real Pythagorean iff the field $k((t))$ is formally real Pythagorean.
c) If k is Euclidean, then the iterated Laurent series field $k((t_1))\cdots((t_n))$ is formally real Pythagorean, has precisely 2^n orderings and precisely 2^{n+1} square classes.

Proof. Let $<$ be an ordering on k. For $x = \sum_{n=N}^\infty a_n t^n \in k((t))^\bullet$, put $x > 0 \iff a_N > 0$. This gives an ordering on $k((t))$ extending the given ordering $<$ on k, and in which $t > 0$. Now let $<$ be an ordering on $k((t))$ extending the given ordering $<$ on k, in which $t > 0$. Then for $x = \sum_{n=N}^\infty a_n t^n \in k((t))^\bullet$, we have

$$x = t^N a_N(1 + a_{N+1}t + \ldots +) = t^n a_N y,$$

say. Then $k[[t]]$ is a complete DVR and $y \in k[[t]]^\times$ such that the image of y in the residue field $k[[t]]/\mathfrak{m} = k$ is $1 \in k^\times$. So by the Local Square Theorem there is $z \in k[[t]]$ such that $y = z^2$. It follows that $\operatorname{sgn}(x) = \operatorname{sgn}(a_N)$ and the ordering $<$ on $k((t))$ is the one specified above. Finally,

$$\sum_n a_n t^n \mapsto \sum_n a_n(-t)^n$$

is a field automorphism of $k((t))$, and the image of an ordering $<$ on $k((t))$ extending the given ordering $<$ on k and in which $t > 0$ under this automorphism is an ordering extending the given ordering $<$ on k in which $t < 0$. A similar argument to the above shows that there is exactly one ordering with these properties.
b) Certainly if $k((t))$ is formally real, then so is its subfield k; conversely, if k is formally real, then part a) shows in particular that $k((t))$ is formally real. (It is not difficult to show this directly.) Now suppose $k((t))$ is formally real and Pythagorean, and let $a, b \in k$. Then there is $c \in k((t))$ such that

$$a^2 + b^2 = c^2.$$

Since $a^2 + b^2 \in k$, the element $c \in k((t))$ is therefore algebraic over k. It follows that $c \in k$ – that is, k is algebraically closed in $k((t))$. This case is especially easy to show directly: by writing $c = \sum_{n=0}^\infty a_n t^n$ and comparing coefficients in the equation $c^2 = a^2 + b^2$ one finds that $a_0^2 = a^2 + b^2$ and $a_n = 0$ for all $n \geq 1$. Thus k is Pythagorean. Conversely, suppose k is formally real Pythagorean. It is enough to show that if $a, b \in k[[t]]^\bullet$, then $a^2 + b^2$ is a square in $k[[t]]$. If a and b are both divisible by t^v then

$$a^2 + b^2 = t^{2v}(\frac{a}{t^v})^2 + (\frac{b}{t^v})^2,$$

so, after reordering a and b if necessary, we reduce to the case in which $a_0 \neq 0$, so $a^2 + b^2 \in k[[t]]^\times$. Then

$$a^2 + b^2 = \left(\sum_n a_n t^n\right)^2 + \left(\sum_n b_n t^n\right)^2 = a_0^2 + b_0^2 + td$$

for some $d \in k[[t]]$. Thus the reduction modulo $\mathfrak{m}$ of $a^2 + b^2$ is $a_0^2 + b_0^2$, which is a square since k is Pythagorean, so by the Local Square Theorem, $a^2 + b^2$ is a square.
c) A Euclidean field is a formally real field with exactly two square classes, hence carries a unique ordering. By part a), it follows by induction that $k((t_1) \cdots ((t_n))$ has exactly 2^n orderings. Now recall that if K is a nondyadic Henselian DVF then K has twice as many square classes as its residue field. From this and induction it follows that $k((t_1)) \cdots ((t_n))$ has 2^{n+1} square classes. □

We now denote by $\mathbb{R}$ any real-closed field. (Of course the most natural choice is the real numbers, but the point is that as far as quadratic forms are concerned, it will not matter.) In this section we study quadratic forms over $\mathbb{R}((t))$. Since $\mathbb{R}$ is Euclidean, by Theorem 23 the field $\mathbb{R}((t))$ is formally real Pythagorean with two orderings and 4 square classes: representatives are given by $1, -1, t, -t$. Thus $\mathbb{R}((t))$ lies in the class of fields for which the structure of the Witt ring was determined in Theorem X.X: we have

$$W(\mathbb{R}((t))) \cong \mathbb{Z}\langle 1\rangle \oplus \mathbb{Z}\langle t\rangle.$$

Here is another way to think about this result: let $<_1$ be the ordering on $\mathbb{R}((t))$ in which $t > 0$ and let $<_2$ be the ordering on $\mathbb{R}((t))$ in which $t < 0$. The total signature map gives a homomorphism.

$$\Sigma : W(\mathbb{R}((t))) \to \mathbb{Z}^2, \ q \mapsto (\sigma_1(q), \sigma_2(q)).$$

Because $\mathbb{R}((t))$ is Pythagorean, Σ is injective. If $q = a\langle 1\rangle + b\langle -1\rangle + c\langle t\rangle + d\rangle - t\rangle$, then

$$\Sigma(q) = (a - b + c - d, a - b - c + d),$$

so

$$\Sigma(W(\mathbb{R}((t))) = \{(x, y) \mid x \equiv y \pmod 2\} \subset \mathbb{Z}^2.$$

This shows again that $(W(\mathbb{R}((t)), +)$ is freely generated by $\langle 1\rangle$ and $\langle t\rangle$.

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