REVIEW NOTES FOR EMAT 233

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These notes are meant to help you study for the final exam. Some notation explained: we sometimes write $f(x, y, z) \equiv 0$, which should be read, "f of x, y, z is *identically* equal to zero," i.e., equal to zero on its entire domain. Also we use the notation $\int_{x_m}^{x_M} f(x)dx$, meaning that the integral extends over the interval $x_m \leq x \leq x_M$: the small "m" stands for **minimum** and the large "M" stands for **maximum**.

1. Vectors

Let $\mathbf{v} = (v_1, v_2, v_3)$, $\mathbf{w} = (w_1, w_2, w_3)$ be vectors in space.

The norm of $\mathbf{v}$ is $||\mathbf{v}|| = \sqrt{v_1^2 + v_2^2 + v_3^2}$.

For nonzero $\mathbf{v}$, a unit vector in the direction of $\mathbf{v}$, denoted $\hat{\mathbf{v}}$, is $\hat{\mathbf{v}} = \frac{\mathbf{v}}{||\mathbf{v}||}$.

The dot product (or inner product, scalar product):

$$\mathbf{v} \cdot \mathbf{w} := v_1 w_1 + v_2 w_2 + v_3 w_3.$$

The dot product formula:

$$\mathbf{v} \cdot \mathbf{w} = ||\mathbf{v}|| ||\mathbf{w}|| \cos \theta,$$

where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$. It follows that $\mathbf{v} \cdot \mathbf{w} = 0 \iff \mathbf{v} \perp \mathbf{w}$.

Cross product (or vector product)

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Cross product formula: the magnitude of the cross product is

$$||\mathbf{v} \times \mathbf{w}|| = ||\mathbf{v}|| ||\mathbf{w}|| \sin \theta;$$

this is also the area of the parallelogram spanned by $\mathbf{v}$ and $\mathbf{w}$. Moreover

$$(\mathbf{v} \times \mathbf{w}) \cdot \mathbf{v} = (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{w} = 0.$$

Thus, if $\mathbf{v}$ and $\mathbf{w}$ do not lie along a single line, $\mathbf{v} \times \mathbf{w}$ is perpendicular to the plane spanned by $\mathbf{v}$ and $\mathbf{w}$. Finally, its orientation is given by the right hand rule.

Projection formula: $\text{proj}_{\mathbf{w}} \mathbf{v} = (\frac{\mathbf{v} \cdot \mathbf{w}}{\mathbf{w} \cdot \mathbf{w}})\mathbf{w}$.

Equations of planes: The equation of the plane passing through $P_0 = (x_0, y_0, z_0)$ with normal vector $\mathbf{n} = (a, b, c)$ is

$$\mathbf{n} \cdot ((x, y, z) - P_0) = a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

Note that this is of the form $ax + by + cz + C$, where $C = ax_0 + by_0 + cz_0$. In general, planes of the form

$$ax + by + cz = C_1,\ ax + by + cz = C_2$$

both have normal vector $\mathbf{n}$, so are parallel.

Comments: You must always remember that $\mathbf{v} \cdot \mathbf{w}$ is a *scalar* (a number!), whereas $\mathbf{v} \times \mathbf{w}$ is a vector in space. Writing things like

$$\text{(WRONG!)} \quad \text{“}\mathbf{v} \cdot \mathbf{w} = (v_1 w_1, v_2 w_2, v_3 w_3)''$$

arouses great consternation in mathematics instructors. Subconsciously we think that if you cannot get this right then you don't understand *any* of the material in the course. Thus you should be especially careful to avoid making this mistake. (There are few other equally serious mistakes that will be pointed out later.)

The dot product could be defined for vectors with any number of components in the same way, but the cross product is pecular to $\mathbb{R}^3$.[1]

2. Parameterized curves

The expression $\mathbf{r}(t) = (x(t), y(t), z(t))$ gives a parameterized curve in $\mathbb{R}^3$; as a special case, taking $z(t) \equiv 0$ we get curves in the plane. Parameterized curves come with a domain – i.e., are defined for some time interval, usually of the form $t_m \leq t \leq t_M$ or $-\infty < t < \infty$.

Important examples:

1) Parametric equations of the line passing through two points P and Q. Put $\mathbf{v} = \overline{PQ} = Q - P$, the vector from P to Q. Then

$$\mathbf{r}(t) = P + t\mathbf{v} = (1 - t)P + tQ.$$

If $P = (x_1, y_1, z_1),\ Q = (x_2, y_2, z_2)$, then

$$\mathbf{r}(t) = (x_1 + (x_2 - x_1)t, y_2 + (y_2 - y_1)t, z_2 + (z_2 - z_1)t).$$

Note that $\mathbf{r}(0) = P,\ \mathbf{r}(1) = Q$; in particular, restricting to $0 \leq t \leq 1$ parameterizes the directed line segment from P to Q.

2) The circle (in the plane) centered at (x_0, y_0) with radius r. The defining equation is $(x - x_0)^2 + (y - y_0)^2 = r^2$; the parametric equations are

$$\mathbf{r}(t) = (x_0 + r\cos t, y_0 + r\sin t);$$

you should check that x and y do satisfy the defining equation. For $0 \leq t \leq 2\pi$ the circle gets traversed once counterclockwise.

3) The ellipse (in the plane) centered at (x_0, y_0) with semiaxes a and b, i.e., with defining equation $\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1$. Then

$$\mathbf{r}(t) = (x_0 + a\cos t, y_0 + b\sin t)$$

[1] Actually, there is a generalized cross product in $\mathbb{R}^n$ for all $n \geq 2$, however its input is $n - 1$ vectors, and $n - 1 = 2$ implies $n = 3$.

for $0 \leq t \leq 2\pi$ traverses the ellipse once counterclockwise.

4) A circular helix centered on the z-axis:

$$\mathbf{r}(t) = (r\cos t, r\sin t, kt)$$

for some constant k.

Calculus of parameterized curves:
Velocity $\mathbf{v}(t)$ $\mathbf{r}'(t) = (x'(t), y'(t), z'(t))$.

Speed $v(t) = ||\mathbf{v}(t)|| = \sqrt{(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2}$.

Unit tangent vector $\mathbf{T}(t) = \mathbf{v}(t)/v(t)$.

Acceleration $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t) = (x''(t), y''(t), z''(t))$.

Arcelngth from t_m to t_M: $s = \int_{t_m}^{t_M} v(t)dt$.

Tangential component of acceleration (as a vector): $\mathbf{a}_T = \operatorname{proj}_{\mathbf{v}(t)} \mathbf{a}(t)$. As a scalar take $\mathbf{a}_T \cdot T$.

Normal component of acceleration (as a vector): $\mathbf{a} - \mathbf{a}_T$.

Curvature: $\frac{||\mathbf{r}'(t) \times \mathbf{r}''(t)||}{||\mathbf{r}'(t)||^3}$

3. Diffferential Calculus on Surfaces

In this course we worked with two kinds of surfaces in three-dimensional space:

1) surfaces given as the graph of a function $z = f(x, y)$.
2) implicitly defined by an equation, or *level surfaces* $F(x, y, z) = C$.

Examples: By considering level curves, you should be able to figure out the rough shape of surfaces of the form

$$z = \pm ax^2 \pm by^2 \pm C$$

or

$$z^2 = \pm ax^2 \pm by^2 \pm C$$

Especially, $x^2 + y^2 + z^2 = R^2$ is a sphere of radius R, and $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is an ellipsoid with semiaxes a, b, c.

For any function $f(x, y)$, can regard y as a constant and differentiate with respect to x, which we write as $\frac{\partial f}{\partial x}$. Similarly by holding x constant and differentiating with respect to y we have $\frac{\partial f}{\partial y}$.

Chain rule: $\frac{d}{dt}(f(x(t), y(t)) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$.

Directional derivatives: let $\hat{u}$ be a unit vector in the plane. We have the notion of the derivative of $z = f(x, y)$ at (x_0, y_0) in the direction $\hat{u}$, as follows: it is the z-component of the velocity vector of the curve

$$R(t) = (x(t), y(t), f(x(t), y(t))),$$

where $\mathbf{r}(t) = (x(t), y(t))$ is any plane curve with $\mathbf{r}(0) = (x_0, y_0), \mathbf{r}'(0) = \hat{u}$. A calculation using the chain rule shows that

$$D_{\hat{u}}(f)(x_0, y_0) = (\frac{\partial f}{\partial x}(x_0, y_0), \frac{\partial f}{\partial y}(x_0, y_0)) \cdot \hat{u}.$$

In particular

$$D_{(1,0)}f = \frac{\partial f}{\partial x},\ D_{(0,1)}f = \frac{\partial f}{\partial y}.$$

The gradient: if $f = f(x, y)$, then $\nabla(f) = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$. Similarly, if $F = F(x, y, z)$, $\nabla(F) = (\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z})$.

Using the dot product formula,

$$D_{\hat{u}}f(x_0, y_0) = \nabla(f)(x_0, y_0) \cdot \hat{u} = ||\nabla f(x_0, y_0)||||\hat{u}|| \cos\theta = ||\nabla(f)(x_0, y_0)|| \cos\theta.$$

This expression is maximized when $\theta = 0$ and minimized when $\theta = \pi$. That is, the largest possible value of a directional derivative at (x_0, y_0) is attained in the direction of $\nabla(f)$: the gradient gives the path of steepest ascent. Similarly, $-\nabla(f)$ gives the path of steepest descent.

Tangent plane to $z = f(x, y)$ at (x_0, y_0): it is the unique plane passing through (x_0, y_0, z_0) and containing all tangent vectors $R'(t_0)$ to curves $R(t) = (x(t), y(t), R(x(t), y(t))$ on S passing through $(x_0, y_0, z_0 = f(x_0, y_0))$. Explicitly, the equation is

$$z = f(x_0, y_0) = (x - x_0)\frac{\partial f}{\partial x}(x_0, y_0) + (y - y_0)\frac{\partial f}{\partial y}(x_0, y_0).$$

Tangent plane to a level surface $F(x, y, z) = C$ at (x_0, y_0, z_0): a normal vector $\mathbf{n}$ is given by $\nabla(F)(x_0, y_0, z_0) = (\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z})$, so the equation of the plane is

$$\frac{\partial F}{\partial x}(x_0, y_0, z_0)(x - x_0) + \frac{\partial F}{\partial y}(x_0, y_0, z_0)(y - y_0) + \frac{\partial F}{\partial z}(x_0, y_0, z_0)(z - z_0) = 0.$$

Note that if $z = f(x, y)$ is a surface of type 1), we can view it as a level surface by defining

$$F(x, y, z) = z - f(x, y) = 0.$$

Then $\nabla(F) = (-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1)$, and the equation of the tangent plane becomes

$$-\frac{\partial f}{\partial x}(x - x_0) - \frac{\partial f}{\partial y}(y - y_0) + (z - z_0) = 0,$$

or

$$z = z_0 + \frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial y}(y - y_0);$$

since $z_0 = f(x_0, y_0)$, this is the same as the previous equation.

4. Vector fields

Let

$$F(x, y, z) = (P(x, y, z), Q(x, y, z), R(x, y, z)) = Pdx + Qdy + Rdz$$

be a vector field in space. (To get a vector field in the plane, take $R \equiv 0$ and P and Q to be only functions of x and y.)

Divergence: $\text{Div}(F) = \nabla \cdot F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$. The physical interpretation of divergence is as flux density: let B be a small box containing the point $P_0 = (x_0, y_0, z_0)$. The flux density is the limit of the flux of F through the surface of B divided by the volume of B as the length of each side of B approaches zero. We showed that this limit equals $\text{Div}(F)(P_0)$.

A vector field is said to be **incompressible** if $\text{Div}(F) \equiv 0$ on its entire domain.

Curl: $\text{curl}(F) = \nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = (\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, -(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}), \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})$.

Special case: if F is a vector field in the plane – $R \equiv 0$ and $P = P(x, y)$, $Q = Q(x, y)$, then $\text{curl}(F) = (0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})$. Indeed, we often write $\text{curl}(F) = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ for planar vector fields, with the understanding that this is a convenient abbrevation for the precise statement:

$$\text{curl}(F) \cdot \hat{\mathbf{k}} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}.$$

Physical interpretation: the curl measures the infinitesimal rotation of F at a point P: if we nail down a paddlewheel at P and orient it so that its axis aligns with $\text{curl}(P)$, then it will turn, in the direction consistent with the righthand rule. In particular, if $\text{curl}(P) \neq 0$ at P, we can buiold a device which extracts "energy" at P, so F, viewed as a field of forces, is not "conservative."

A vector field is said to be **irrotational** if $\text{curl}(F) = 0$ on its entire domain.

Two identities: if $f(x, y, z)$ is a scalar-valued function, then $\nabla(f)$ is a vector field, called a gradient field.

$$\text{curl}(\nabla(f)) \equiv \hat{0}.$$

Similarly,

$$\text{Div}(\text{curl}(F)) \equiv 0.$$

That is, gradient vector fields are irrotational, and "curl vector fields" are incompressible.

Singularities: many naturally occuring vector fields have **singularities**, i.e., they are not defined (and could not be defined in a continuous manner) on all of $\mathbb{R}^3$.

Example: The Newton/Coulomb field $F(x, y, z) = K\frac{\mathbf{r}}{||\mathbf{r}||^3}$. This is defined everywhere except the origin.

Example: $F(x, y, z) = (\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, z)$. This is defined except where $x = y = 0$, i.e., everywhere except the z-axis.

5. Multiple integrals

Using Riemann sums we can define the integral of a scalar function over a region R of the plane or a region V in space. But, just as for single integrals, we do not dare to evaluate any but the simplest of functions using this definition. Instead, we write the double integral as an iterated integral and use the fundamental theorem of calculus.

Simplest case R is a rectangle $x_m \leq x \leq x_M,\ y_m \leq y \leq y_M$. Then

$$\int\int_R f(x, y)dA = \int_{y=y_m}^{y=y_M} \int_{x=x_m}^{x=x_M} f(x, y)dxdy = \int_{x=x_m}^{x=x_M} \int_{y=y_m}^{y=y_M} f(x, y)dydx.$$

That is, the nested single integrals in each order are equal and both are equal to the double integral as defined using Riemann sums.

In three variables: if V is box: $x_m \leq x \leq x_M,\ y_m \leq y \leq y_M,\ z_m \leq z \leq z_M$.

$$\int\int\int_V f(x, y, z)dV = \int_{z_m}^{z_M} \int_{y_m}^{y_M} \int_{x_m}^{x_M} f(x, y, z)dxdydz,$$

and similarly for the five other possible orders: $dxdzdy$, $dydxdz$, $dydzdx$, $dzdydx$.

The plot thickens when we want to integrate over regions with curved boundaries. For instance, to integrate over the unit circle, imagine integrating first with respect to x:

$$\int_{y=-1}^{y=1} \int_{x=?}^{x=??} f(x, y)dxdy.$$

The point is that how far to the left and to the right the region extends *depends* on the y-coordinate: e.g. at $y = \pm 1$ we have just $x = 0$; when $y = 0$, x needs to go from -1 to 1. So the inner limits must be functions of y: indeed $x_M = C_R(y),\ x_m = C_L(y)$, where $C_L(y)$ and $C_R(y)$ are functions of y giving the left and right boundaries of the region R. In our case the curve is defined by $x^2 + y^2 = 1$, so $C_R = \sqrt{1-y^2},\ C_L = -\sqrt{1-y^2}$, and the integral is

$$\int_{y=-1}^{y=1} \int_{x=-\sqrt{1-y^2}}^{x=\sqrt{1-y^2}} f(x, y)dxdy.$$

Formally speaking the outer limits – which are always constants – are given by the *projection* of the region R into the y-axis (in the $dxdy$ case) into the x-axis (in the $dydx$ case).

Also, an assumption has implicitly been made about the shape of the region R: to integrate $\int_{y_m}^{y_M} \int_{x=C_L(y)}^{x=C_R(y)} f dxdy$, we need R to have the property that if (x_1, y) and (x_2, y) are both in R, then so is the entire (horizontal) line segment connecting them. We call this property of a region **x-convex**. Similarly, integrating with respect to $dydx$ in the way we did assumes that the region is **y-convex**: if (x, y_1) and (x, y_2) are in R, then so is the entire (vertical) line segment connecting them.

As an example of a region that is neither x-convex nor y-convex, consider the region R between the two circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$. What do we do to evalute $\int\int_R f(x, y)dA$? There are two possibilities: fist, we could split R into finitely many subregions, each of which is either x-convex or y-convex. Indeed, any region can be divided into finite many subregions each of which is both x-convex and y-conex by "cutting" along horiztonal and vertical lines.[2] In the present example, the x- and y-axes do the trick, cutting R into four arch-shaped sectors.

If $f(x, y)$ is defined not just on R but on all of $x^2 + y^2 \leq 4$, we may write $R = D_2 - D_1$, where D_2 and D_1 are the larger disk (of radius 2) and the smaller disk (of radius 1) respectively. Then

$$\int\int_R f(x, y)dA = \int\int_{D_2} f(x, y)dA - \int\int_{D_1} f(x, y)dA.$$

A similar, albeit more geometrically challenging, discussion holds for triple integrals. For instance, supposing that V is a space region, to set up $\int\int\int_V f(x, y, z)dzdydx$, the linner limits are $z_m = C_B(x, y),\ z_M = C_T(x, y)$, the equations of the surfaces which bound V above and below. (This assumes that V is z-convex: if (x, y, z_1) and (x, y, z_2) are both in V, then so is the entire line segment joining them.) To get the remaining limits we consider the *projection* R of V into the xy-plane, and parameterize R as above.

Example: Let V be the region bounded by the sphere of radius r. Then $z = \pm\sqrt{r^2 - x^2 - y^2}$, so $z_m = -\sqrt{r^2 - x^2 - y^2}$, $z_M = \sqrt{r^2 - x^2 - y^2}$. The projection R of V into the xy-plane is the disk of radius r, so similarly to the above example is parameterized as $y_m = -\sqrt{r^2 - x^2},\ y_M = \sqrt{r^2 - x^2},\ x_m = -r,\ x_M = r$. Thus

$$\int\int\int_V f(x, y, z)dV = \int_{x_m=-r}^{x_M=r} \int_{y_m=-\sqrt{r^2-x^2}}^{y_M=\sqrt{r^2-x^2}} \int_{z_m=-\sqrt{r^2-x^2-y^2}}^{z_M=\sqrt{r^2-x^2-y^2}} f(x, y, z)dzdydx.$$

Other coordinate systems: on the exam you will certainly be asked to do at least one integral by changing to either polar, cylindrical or spherical coordinates. The formulas you need to remember are listed below. You should make sure to practice parameterizing regions in polar and spherical coordinates.

Polar coordinates:

$$r = \sqrt{x^2 + y^2},\ \theta = \tan^{-1}(y/x).$$
$$x = r\cos\theta, y = r\sin\theta.$$
$$dA = rdrd\theta.$$

Cylindrical coordinates: r and θ are related to x and y as above.

$$dV = rdrd\theta dz.$$

Spherical coordinates:

$$\rho = \sqrt{x^2 + y^2 + z^2}, \theta = \tan^{-1}(y/x), \varphi = \cos^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right).$$
$$x = \rho\cos\theta\sin\varphi,\ y = \rho\sin\varphi\sin\theta,\ z = \rho cos\varphi.$$
$$dV = \rho^2 \sin\varphi\ d\rho\ d\theta\ d\varphi.$$

[2] At least, this is true for all regions we will ever meet in this course.

6. Line integrals

Line integral of a scalar function: Let $\mathbf{r}(t) = (x(t), y(t), z(t))$ be a parameterized curve, defined for $t_m \leq t \leq t_M$. Let $g(z, y, z)$ be a scalar function defined along the curve. We define

$$\int_C g ds := \int_{t_m}^{t_M} f(x(t), y(t), z(t)) \frac{ds}{dt} dt = \int_{t_m}^{t_M} f(x(t), y(t), z(t)) \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2} dt.$$

This is indepednent of the chosen parameterization of C, and even of the orientation.

Line integral of a vector field along a curve: let $F = (P, Q, R)$ be a vector field defined along the parameterized curve $\mathbf{r}(t)$.

$$\int_C F \cdot d\mathbf{r} = \int_{t_m}^{t_M} F(x(t), y(t), z(t)) \cdot \mathbf{r}'(t) dt =$$

$$\int_{t_m}^{t_M} P(x(t), y(t), z(t)) x'(t) + Q(x(t), y(t), z(t)) y'(t) + R(x(t), y(t), z(t)) z'(t) dt.$$

Notice that we are integrating the tangential component of F along the curve: that is, the line integral may interpreted as the work done in moving a particle through a field of forces F along the path C.

The line integral of a vector field is independent of the parameterization except that it depends upon the **orientation** of C: if we traverse C in the opposite direction, $\mathbf{r}'(t)$ changes sign, and the line integral would change by a factor of -1. Thus we now write C meaning an **oriented** curve – i.e., traversed in a fixed direction (this is equivalent to specifying a choice of unit vector on the tangent line to C at each point). If we wish to traverse the same curve but in the opposite direction, we use the notation $-C$. In particular, we have

$$\int_{-C} F \cdot d\mathbf{r} = -\int_C F \cdot d\mathbf{r}.$$

If $\mathbf{r}(t_m) = \mathbf{r}(t_M)$ – i.e., if the initial and terminal points coincide – C is said to be **closed** (there are, of course, still two orientations!), and one sometimes writes $\oint_C F \cdot d\mathbf{r}$ for line integrals around closed curves. This notation is really ornamental: nothing would change if we didn't include it. (A similar comment holds for $\frac{\partial f}{\partial x}$: if we wrote instead $\frac{df}{dx}$, what else could possibly be meant?)

7. Conservative vector fields and the Fundamental Theorem

A vector field F is **conservative** if for any closed curve C in the domain of F, $\oint_C F \cdot d\mathbf{r} = 0$.[3]

A vector field F is **path-independent** if for any two oriented curves C_1 and C_2 in the domain of F with common initial point P and terminal point Q, $\int_{C_1} F \cdot d\mathbf{r} = \int_{C_2} F \cdot d\mathbf{r}$. For a path-independent vector field, it makes sense to write $\int_P^Q F \cdot d\mathbf{r}$ for the line integral of F from any initial point P to any terminal point Q.

[3]It suffices for the line integral to be zero for all **simple** closed curves, i.e., without self-intersection. We will usually work with simple curves without further comment.

A vector field F is a **gradient field** if there exists a scalar-valued function $f(x, y, z)$ defined on the entire domain of F such that $\nabla f = F$.

Theorem 1. *(Fundamental Theorem of Calculus for Line Integrals) The properties of being conservative, path-independent and a gradient field are all equivalent: when any one holds, so do both of the others. Moreover, when they do hold, we can use the function f to evaluate the line integrals as in one-variable calculus:*

$$\int_P^Q F \cdot d\mathbf{r} = f(Q) - f(P).$$

From the section on vector fields, we saw that $\mathrm{curl}(\nabla(f)) = \overline{0}$; that is, conservative vector fields are irrotational. (Recall that we argued for this physically, in terms of "energy" being gained or lost.)

This provides a way to show that a vector field F is **not** conservative: namely, show that its curl is nonzero.

Warning: If F has singularities, it is **NOT** necessarily true that $\mathrm{curl}(F) = \overline{0}$ implies F is conservative.

Example: Let $F(x, y, z) = (\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, z)$. We remarked in Section 5 that F has singularities along the z-axis. Moreover, you should check that $\mathrm{curl}(F) \equiv 0$ for all points at which F is defined (i.e., except when $x = y = 0$). On the other hand, let C be the unit circle in the xy-plane ($z = 0$) oriented counterclockwise. Drawing a picture of the vector field indicates that for every point on the circle F points in the same direction as the tangent vector, which means that the line integral must be positive. A calculation shows $\oint_C F \cdot d\mathbf{r} = 2\pi$, which is not zero. **Therefore F is an irrotational vector field which is not conservative.**

Advice: If you are asked, "Show that F is conservative," the best way to go is to explicitly find the function f such that $\nabla f = F$: there is a procedure for this involving repeated integration and differentiation: see page 500 of your text. Checking that $\mathrm{curl}(F) = 0$ may not be sufficient, as the previous example shows; more practically, it is likely that the next part of the question will be to compute the line integral of F along any curve from P to Q, and for this you will probably want to use the fundamental theorem anyway.

8. Green's Theorem

In this section everything takes place in the plane.

Normal line integrals in the plane: if C is a simple closed curve in the plane, it has a natural orientation in which we keep the bounded region R which it encloses on our left. Accordingly, we can define on any point of C an **outward unit normal** $\hat{n}$, which is always 90 degrees to the right of the tangent vector to C in the natural (counterclockwise) orientation. Usually we have been integrating $F \cdot \mathbf{v}$, the tangential component of the vector field. But nothing stops us from integrating the normal component instead: indeed the **normal line integral**

$$\oint_C F \cdot \hat{n} ds$$

is precisely giving the flux of F through C, the boundary of the region R. Since we saw that the divergence equals the flux density, and in general the integral of "something density" should be "something" (this is what density means), this (at least) suggests the following result.

Theorem 2. *(Green's Theorem in Divergence Form) Let R be a region in the plane whose boundary is C. Then, with orientations as discussed above,*

$$\int\int_R \mathrm{Div}(F) dA = \oint_C F \cdot \hat{n} dS.$$

By formal manipulations (involving turning of vector fields), this version of Green's Theorem can be shown to be equivalent to the following version:

Theorem 3. *(Green's Theorem, Curl Form) Let R be a plane region bounded by a simple closed curve C. Then*

$$\int\int_R \mathrm{curl}(F) \cdot \hat{\mathbf{k}} dA = \int\int_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA = \oint_C F \cdot d\mathbf{r}.$$

A generalization: Green's Theorem continues to hold for planar regions R which have several boundary components, provided they are oriented appropriately – we traverse each boundary component so that the interior of R is on our left. We write ∂R for the boundary of R. Then

$$\int\int_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA = \oint_{\partial R} F \cdot d\mathbf{r}.$$

This general version is useful in dealing with singularities of vector fields: suppose that F has a singularity at P (and, for simplicity, nowhere else). Let C_1 and C_2 be two simple closed curves oriented counterclockwise, each enclosing P, and such that C_1 is in the interior of C_2. Both C_1 and C_2 bound regions of the plane, but Green's Theorem **cannot** be applied directly to either of these regions, since F has a singularity at P which is contained in both. However, let R be the annular (= ring-shaped; think *Seigneur d'anneaux*) region *between* C_1 and C_2. Here $\partial R = C_2 - C_1$, by which we mean that in order to get the boundary of R on our left we traverse C_2 counterclockwise but C_1 *clockwise*. Then Green's Theorem applies on R, and says

$$\int\int_R \mathrm{curl}(F) dA = \int_{C_2 - C_1} F \cdot d\mathbf{r} = \oint_{C_2} F \cdot d\mathbf{r} - \oint_{C_1} F \cdot d\mathbf{r}.$$

For example: suppose that F is irrotational: $\mathrm{curl}(F) \equiv \overline{0}$. Then the left-hand side is zero, and we are getting that $\oint_{C_1} F \cdot d\mathbf{r} = \oint_{C_2} F \cdot d\mathbf{r}$. That is, all line integrals of F along simple closed curves winding counterclockwise around P have the same value, so in order to evaluate any one of them we can choose whichever closed curve makes the computation easiest. This was the key to Problem 3d) on the second midterm, which was taken from #26 in Section 9.12.

The Area Formula: Let $F = \frac{1}{2}(-ydx + xdy)$. Note $\mathrm{curl}(F) = 1$. Let R be a plane region bounded by a simple closed curve C. By Green's Theorem we get a formula for the area of R, namely

$$\mathrm{area}(R) = \int\int_R 1 dA = \int\int_R \mathrm{curl}(F) dA = \oint_C \frac{1}{2}(-ydx + xdy).$$

9. Surface integrals, Stokes' Theorem, the Divergence Theorem

Let S be the graph of $z = f(x, y)$ over some planar region R. We computed the surface area element dS; it is

$$dS = \sqrt{1 + (\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2} dA.$$

Thus the surface area of S is

$$\int\int_S 1 dS = \int\int_R \sqrt{1 + (\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2} dA.$$

This allows us to integrate scalar-valued functions $g(x, y, z)$ defined on the surface S:

$$\int\int_S g dS = \int\int_R g(x, y, f(x, y)) \sqrt{1 + (\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2} dA.$$

We are more interested in integrating vector fields: if $F = (P, Q, R)$ is a vector field defined on a surface S, we want a notion of a surface integral – which should be compared to the **normal** line integral along a curve rather than the more conventional tangential line integral. In particular, if S is a closed surface, then we want the surface integral of F over S to compute the flux of F through S.

We need an **orientation** of S, which is a consistent (continuous) choice of unit normal vector $\hat{n}$ at each point of S. (For some surfaces – such as the Mobius strip – no such consistent choice of normal vector is possible. However, since we want to do surface integrals we do not meet such surfaces in this course.)

If $(S, \hat{n})$ is a surface together with a choice of normal vector, then the surface integral of F along S is by definition

$$\int\int_S F \cdot \hat{n} dS = \int\int_R F \cdot \hat{n} \sqrt{1 + (\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2} dA.$$

The case of interest to us is when S is given as the graph of $z = f(x, y)$ over R, there is a preferred choice of normal, the **upward normal**. Define $\mathbf{N} = (1, 0, \frac{\partial f}{\partial x}) \times (0, 1, \frac{\partial f}{\partial y}) = (-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1)$. (Compare with the discussion on page 4: this really does give a normal vector to the surface: since the z-component is always one, it points in an upward direction rather than a downward direction.) As a bonus, we have $||\mathbf{N}|| = \sqrt{1 + (\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2} = \frac{dS}{dA}$, so that $\hat{n} = \mathbf{N}/||\mathbf{N}||$, but $\mathbf{N}$ is actually more useful than $\mathbf{n}$:

$$\int\int_S F \cdot \hat{n} dS = \int\int_S F \cdot \mathbf{N}/||\mathbf{N}|| ||\mathbf{N}|| dA = \int\int_R F(x, y, f(x, y)) \cdot (-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1) dA.$$

As with line integrals, if we switched the orientation by changing $\hat{n}$ to $-\hat{n}$, the surface integral would be multiplied by a factor of -1.

Theorem 4. *(Stokes' Theorem) Let $(S, \hat{n})$ be an oriented surface in $\mathbb{R}^3$ with boundary curve(s) ∂S. We orient each boundary component so that as we walk along the* *top of the surface S – as specified by $\hat{n}$ – the interior of S is on our left. Then*

$$\int\int_S \mathrm{curl}(F) \cdot \hat{n} dS = \int_{\partial S} F \cdot d\mathbf{r}.$$

Notice that if $S = R$ is a planar region, then $dS = dA$, $\hat{n} = \hat{\mathbf{k}}$, and we recover Green's Theorem. (Conversely, one way to establish Stokes' Theorem is to divide the surface into very small pieces, argue that each piece is approximately planar, and apply Green's Theorem.)

Theorem 5. *(Gauss' Theorem aka The Divergence Theorem) Let V be a region in space with boundary surface(s) ∂V, each oriented via outward normals. Then*

$$\int\int\int_V \mathrm{Div}(F) dV = \int\int_{\partial V} F \cdot \hat{n} dS.$$

An application of Stokes' theorem: let F be a vector field defined on a region V of $\mathbb{R}^3$ enjoying the following property: every simple closed curve C contained in V is the boundary of at least one surface S contained entirely in V. We claim that on such a V, every irrotational vector field is conservative.

Indeed, let C be the simple closed curve, let S be a surface such that $C = \partial S$, and apply Stokes' Theorem to S: we get

$$\oint_C F \cdot d\mathbf{r} = \int\int_S \mathrm{curl}(F) \cdot \hat{n} dS = \int\int_S 0 \cdot \hat{n} dS = \int\int 0 dS = 0.$$

The condition that every simple closed curve in V bounds a surface is true when V is all of three-dimensional space, i.e., for vector fields with no singularities. More generally, it is satisfied when the domain V of F is **simply connected**: that is, when any lasso in V can be contracted to a point without leaving V. This is not exactly the same as saying that V has "no holes": indeed, if F has only finitely many singularities, then this will not stop us from contracting our lasso: we can bring it over or under any isolated points. On the other hand, consider again the vector field

$$F(x, y, z) = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2}, z\right).$$

This F is not defined when $x = y = 0$, i.e., along the entire z-axis. The unit circle C in the xy-plane gives an example of a lasso which cannot be contracted to a point in V: it "gets caught" on the z-axis. Thus the domain of F is *not* simply connected. Moreover, we saw that $\oint_C F \cdot d\mathbf{r} = 2\pi \neq 0$, so that F is **NOT** conservative: this shows that the unit circle C is not the boundary of any surface S which does not meet the z-axis.[4]

In summary, depending upon the "shape" of the domain V, it may or may not be the case that every irrotational vector field defined on all of V is conservative: it is safest to actually find an explicit function f such that $\nabla f = F$.

[4]In fact this gives a very strange *proof* that no such surface exists: we deduce this very geometric fact from a certain integral being 2π and not zero! The idea of turning this argument around and studying the "topological" properties of a region V via integrals of vector fields on this region was pursued and vastly generalized in the twentieth century by **Gustave de Rham**; the resulting field, known now as de Rham cohomology, is a major branch of contemporary mathematics.