THE CARDINAL KRULL DIMENSION OF A RING OF HOLOMORPHIC FUNCTIONS

PETE L. CLARK

Abstract. We give an exposition of a recent result of M. Kapovich on the cardinal Krull dimension of the ring of holomorphic functions on a connected $\mathbb{C}$-manifold. By reducing to the one-dimensional case we give a stronger lower bound for Stein manifolds.

1. Introduction

Recently Kapovich proved the following striking result: if a connected $\mathbb{C}$-manifold M has a noncontant holomorphic function, then the ring $\operatorname{Hol}(M)$ of holomorphic functions on M has a chain of prime ideals of length the continuum $\mathfrak{c} = 2^{\aleph_0}$ [Ka15].

Here we give an exposition of Kapovich's Theorem, in which the algebraic part of the proof is shortened and simplified. Kapovich defines a class of "ample rings" and shows – by a beautiful use of Sard's Theorem that we follow closely – that if $\operatorname{Hol}(M) \supsetneq \mathbb{C}$ then $\operatorname{Hol}(M)$ is ample, and then he shows that any ample ring has a chain of prime ideals of continuum length [Ka15, Thm. 4]. The proofs, and even the definition of ample rings, make use of hyperreals and hypernatural numbers.

We observe that the analytic part of Kapovich's argument shows that if $\operatorname{Hol}(M) \supsetneq \mathbb{C}$ then $\operatorname{Hol}(M)$ admits an infinite sequence of discrete valuations $\{v_k\}$ which are independent in the sense that for any sequence $\{n_k\}$ of natural numbers there is $f \in \operatorname{Hol}(M)$ such that $v_k(f) = n_k$ for all k, and then we show (Theorem 1.4) that any ring admitting a sequence of independent discrete valuations has a chain of prime ideals of continuum length. In place of nonstandard analysis our proofs use *ultralimits*. In order to understand these, a reader need only know what an ultrafilter is and that an ultrafilter on a compact space converges to a unique point.

Although $\mathfrak{c}$ is a large number, it may not be large enough! Results of Henriksen [He53] and Alling [Al63] show that if M is a noncompact Riemann surface then $\operatorname{Spec}\operatorname{Hol}(M)$ has a chain of length $2^{\aleph_1}$. Building on these results we show (Theorem 3.3) that for a connected $\mathbb{C}$-manifold M, if $M \cong V \times N$ for a Stein manifold V of positive dimension, then $\operatorname{Hol}(M)$ admits a chain of prime ideals of length $2^{\aleph_1}$.

1.1. **Krull dimensions of partially ordered sets and topological spaces.**

Throughout, all rings are commutative and with multiplicative identity. For a ring R, $\operatorname{Spec} R$ is the set of prime ideals of R, partially ordered under inclusion.

A **chain** is a linearly ordered set; its **length** is its cardinality minus one. The **cardinal Krull dimension** $\operatorname{carddim} X$ of a partially ordered set X is the supremum of lengths of its chains. For a ring R we put $\operatorname{carddim} R = \operatorname{carddim} \operatorname{Spec} R$.

The prime spectrum $\operatorname{Spec} R$ of a ring is endowed with the Zariski topology, in which the closed sets are $V(I) = \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{p} \supset I\}$ as I ranges over all ideals of R. For $\mathfrak{p}_1, \mathfrak{p}_2 \in \operatorname{Spec} R$ we have $\mathfrak{p}_1 \subset \mathfrak{p}_2 \iff \mathfrak{p}_2 \in \overline{\{\mathfrak{p}_1\}}$. Thus $\operatorname{carddim} R$ is a topological invariant of $\operatorname{Spec} R$.

For a topological space X, define the **cardinal Krull dimension** $\operatorname{carddim} X$ as the supremum of lengths of chains of closed irreducible subspaces of X. Since for a ring R, the map $\mathfrak{p} \mapsto V(\mathfrak{p})$ gives an antitone bijection from $\operatorname{Spec} R$ to the set of closed irreducible subspaces of $\operatorname{Spec} R$, we have $\operatorname{carddim} R = \operatorname{carddim} \operatorname{Spec} R$.

Our use of "cardinal" is twofold: (i) it is common to say "$\dim R$ is infinite" if there are arbitrarily long finite chains in $\operatorname{Spec} R$. For the class of rings considered here we will show that the Krull dimension is always zero or infinite, and we will also address (though not completely answer) the more refined question of how infinite it is. (ii) There is also a notion of **ordinal Krull dimension** of rings [GoRo] that we do not discuss here.

Remark 1.1.
a) Let X and Y bre partially ordered sets. If there is an injective isotone map $\iota : Y \to X$, then $\operatorname{carddim} Y \leq \operatorname{carddim} X$.
b) If $f : R_1 \to R_2$ is surjective or a localization map, then $f^* : \operatorname{Spec} R_2 \to \operatorname{Spec} R_1$ is an injective isotone map, so $\operatorname{carddim} R_2 \leq \operatorname{carddim} R_1$.

1.2. Holomorphic functions on a $\mathbb{C}$-manifold.

Let M be a $\mathbb{C}$-manifold. (Our definition includes that M is Hausdorff and second countable.) Let $\operatorname{Hol}(M)$ be the ring of global holomorphic functions $f : M \to \mathbb{C}$. We have $\mathbb{C} \hookrightarrow \operatorname{Hol}(M)$ via the constant functions.

Lemma 1.2. *The ring $\operatorname{Hol}(M)$ is a domain iff M is connected.*[1]

Proof. If $M = M_1 \coprod M_2$ with $M_1, M_2 \neq \varnothing$, let f_i be the characteristic function of M_i. Then $f_1, f_2 \in \operatorname{Hol}(M) \setminus \{0\}$ and $f_1 f_2 = 0$.

Conversely, let $f \in \operatorname{Hol}(M) \setminus \{0\}$ and let U be the set of $x \in M$ such that the power series expansion at x is zero (as a formal series: i.e., every term is zero). For all $x \in U$, f vanishes identically in some neighborhood of x, so U is open. If $x \in M \setminus U$, then some mixed partial derivative of f is nonvanishing at x. These mixed partials are continuous, so there is a neighborhood N_x of x on which this condition continues to hold, and thus $N_x \subset M \setminus U$ and U is closed. Since M is connected and $U \subsetneq M$, we have $U = \varnothing$. For $f, g \in \operatorname{Hol}(M) \setminus \{0\}$, let $x \in M$. The power series of f and g at x are each nonzero, hence the same holds for fg. So fg does not vanish identically on any neighborhood of x: thus $fg \neq 0$. $\square$

From now on we will assume that all our $\mathbb{C}$-manifolds are connected.

1.3. Kapovich's Theorems: Statements.

Theorem 1.3. *(Kapovich [Ka15]) Let M be a $\mathbb{C}$-manifold such that $\operatorname{Hol}(M) \supsetneq \mathbb{C}$. Then $\operatorname{Spec} \operatorname{Hol}(M)$ admits a chain of length $\mathfrak{c} = 2^{\aleph_0}$. Thus* $\operatorname{carddim} \operatorname{Hol}(M) \geq \mathfrak{c}$.

A **discrete valuation** on a ring R is a surjective function

$$v : R \to \mathbb{N} \cup \{\infty\}$$

[1] Our convention is that the empty topological space is not connected.

such that
(DV0) For all $x \in R$, $v(x) = \infty \iff x = 0$.
(DV1) For all $x, y \in R$, $v(xy) = v(x) + v(y)$.
(DV2) For all $x, y \in R$, $v(x + y) \geq \min v(x), v(y)$.

Here we use standard conventions on arithmetic in the extended real numbers: for all $x \in [0, \infty]$, $x + \infty = \infty$ and $\min(x, \infty) = x$. Conditions (DV0) and (DV1) ensure that a ring admitting a discrete valuation is a domain. A V_∞**-ring** is a ring R admitting a sequence $\{v_k\}_{k \in \mathbb{Z}^+}$ of discrete valuations such that for any sequence $\{n_k\}_{k=1}^\infty$ of natural numbers there is $x \in R \setminus \{0\}$ such that $v_k(x) = n_k$ for all $k \in \mathbb{Z}^+$. The following results together imply Theorem 1.3. We give the proofs in §2.

Theorem 1.4. *If R is a V_∞-ring, then* $\operatorname{carddim} R \geq \mathfrak{c}$.

Theorem 1.5. *Let M be a $\mathbb{C}$-manifold. If M admits a nonconstant holomorphic function, then* $\operatorname{Hol}(M)$ *is a V_∞-ring.*

2. Kapovich's Theorems: Proofs

2.1. **Preliminaries on ultralimits.**

Let I be a set, X a topological space, and $x_\bullet : I \to X$ be a function. Let $\mathcal{F}$ be an ultrafilter on I. We say $x \in X$ is an **ultralimit** of $x_\bullet$ and write $\mathcal{F} \lim x_\bullet = x$ if $x_\bullet(\mathcal{F}) \to x$: that is, for every neighborhood U of $x \in X$, we have $x_\bullet^{-1}(U) \in \mathcal{F}$.

Remark 2.1. *From the general theory of filter convergence (e.g.* [Cl-C]*) we deduce:*
(i) If X is Hausdorff, an I-indexed sequence $x_\bullet : I \to X$ has at most one ultralimit.
(ii) If X is quasi-compact, every I-indexed sequence has at least one ultralimit.
(iii) If X is compact, every I-indexed sequence has a unique ultralimit.

In our application we will have $I = \mathbb{N}$, ω a fixed nonprincipal ultrafilter and $X = [0, \infty]$. Thus we have an ordinary sequence $\{x_k\}$ in $[0, \infty]$, and $\omega \lim x_k = x$ means: for all $\epsilon > 0$, the set of $k \in \mathbb{N}$ such that $|x_k - x| < \epsilon$ lies in ω. Because $[0, \infty]$ is compact, any sequence in $[0, \infty]$ has a unique ultralimit.

Remark 2.2. *Let ω be a nonprincipal ultrafilter on $\mathbb{Z}^+$.*
a) If $\lim_{k \to \infty} x_k = x$ in the usual sense, then also $\omega \lim x_k = x$.
b) Let $\{x_k\}$, $\{y_k\}$ be sequences in $[0, \infty]$. Then:
(i) $\omega \lim (x_k + y_k) = \omega \lim x_k + \omega \lim y_k$.
(ii) $\omega \lim \min(x_k, y_k) = \min(\omega \lim x_k, \omega \lim y_k)$.
(iii) $\omega \lim \max(x_k, y_k) = \max(\omega \lim x_k, \omega \lim y_k)$.

2.2. **Proof of Theorem 1.4.**

For $t \in (0, \infty)$, put $\mathfrak{p}_t = \{x \in R \setminus \{0\} \mid \omega \lim_k \frac{v_k(x)}{k^t} > 0\}$. Each $\mathfrak{p}_t$ is a prime ideal, and for all $t_1 \geq t_2$ we have $\mathfrak{p}_{t_1} \subset \mathfrak{p}_{t_2}$. Since R is a V_∞-ring, there is $x_t \in R \setminus \{0\}$ such that $v_k(x_t) = \lceil k^t \rceil$ for all $k \in \mathbb{Z}^+$, and we have $x_t \in \mathfrak{p}_t$, $x_t \notin \mathfrak{p}_s$ for all $s > t$. So $\{\mathfrak{p}_t \mid t \in (0, \infty)\}$ is a chain of prime ideals of R of cardinality $\mathfrak{c}$.

2.3. **Proof of Theorem 1.5.**

Let $h : M \to \mathbb{C}$ be holomorphic and nonconstant. By the Open Mapping Theorem, $U = h(M)$ is a connected open subset of $\mathbb{C}$. In particular U is metrizable and not compact, so there is a sequence $\{z_k\}_{k=1}^{\infty}$ of distinct points of U with no limit point in U. We do not disturb the latter property by successively replacing each z_k with any point in a sufficiently small open ball, so by Sard's Theorem we may assume that each z_k is a regular value of h. For $k \in \mathbb{Z}^+$, let $p_k \in h^{-1}(z_k)$ and let $v_k : \mathrm{Hol}(M) \setminus \{0\} \to \mathbb{N}$ be the order of vanishing of h at p_k: that is, the least N such that there is a mixed partial derivative of order N which is nonvanishing at p_k. Then v_k is a discrete valuation. Let $\{n_k\}_{k=1}^{\infty}$ be as sequence of natural numbers. By the Weierstrass Factorization Theorem [Ru87, Thm. 15.11], there is $g \in \mathrm{Hol}(U)$ such that $\mathrm{ord}_{z_k}(g) = n_k$ and thus – since p_k is a regular value for h – for all $k \in \mathbb{Z}^+$ we have $v_k(g \circ h) = n_k$.

3. The cardinal Krull dimension of a Stein manifold

We will prove a stronger lower bound on the cardinal Krull dimension of $\mathrm{Hol}(M)$ when M is a **Stein manifold**: a $\mathbb{C}$-manifold which admits a closed (equivalently proper) holomorphic embedding into $\mathbb{C}^N$ for some $N \in \mathbb{Z}^+$. Stein manifolds play the role in the biholomorphic category that affine varieties play in the algebraic category (of quasi-projective varieties $V_{/\mathbb{C}}$, say) – and a nonsingular affine variety over $\mathbb{C}$ is a Stein manifold. That is, the Stein manifolds are the $\mathbb{C}$-manifolds which have "enough" global holomorphic functions: in particular, for points $x \neq y$ on a Stein manifold M, there is $f \in \mathrm{Hol}(M)$ with $f(x) \neq f(y)$. At the other extreme lie the compact $\mathbb{C}$-manifolds, which play the role in the biholomorphic category that projective varieties play in the algebraic category – and a nonsingular projective variety over $\mathbb{C}$ is a compact $\mathbb{C}$-manifold. In complex dimension one this is a simple dichotomy: a Riemann surface is a Stein manifold iff it is noncompact [GuRo, p. 209]. However, if M is a compact $\mathbb{C}$-manifold of complex dimension at least 2 and $x \in M$, then $M^\circ = M \setminus \{x\}$ is a noncompact $\mathbb{C}$-manifold with $\mathrm{Hol}\, M^\circ = \mathbb{C}$.

Theorem 3.1. *(Henriksen-Alling) If S, T are noncompact Riemann surfaces then*

$$\mathrm{carddim}\,\mathrm{Hol}(S) = \mathrm{carddim}\,\mathrm{Hol}(T) \geq 2^{\aleph_1}.$$

Proof. Henriksen showed $\mathrm{Spec}\,\mathrm{Hol}(\mathbb{C})$ admits a chain of length $2^{\aleph_1}$ [He53, Thm. 5]. For noncompact Riemann surfaces S and T, Alling showed $\mathrm{Spec}\,\mathrm{Hol}(S)$ and $\mathrm{Spec}\,\mathrm{Hol}(T)$ are homeomorphic [Al79, Thm. 2.14], and as in §1.1 it follows that

$$\mathrm{carddim}\,\mathrm{Hol}(S) = \mathrm{carddim}\,\mathrm{Hol}(\mathbb{C}) \geq 2^{\aleph_1}. \qquad \square$$

Lemma 3.2. *Let M, N be $\mathbb{C}$-manifolds. Then*

$$\mathrm{carddim}\,\mathrm{Hol}(M \times N) \geq \mathrm{carddim}\,\mathrm{Hol}(M).$$

Proof. Let $y_0 \in N$. Pulling back holomorphic functions via the embedding

$$\iota : M \hookrightarrow M \times N, \ x \mapsto (x, y_0)$$

gives a ring homomorphism $\iota^* : \mathrm{Hol}(M \times N) \to \mathrm{Hol}(M)$. If $f \in \mathrm{Hol}(M)$, we put

$$F : M \times N \to \mathbb{C}, \ (x, y) \mapsto f(x).$$

Then $F \in \mathrm{Hol}(M \times N)$ and $\iota^*(F) = f$. So we may apply Remark 1.1b). $\square$

Theorem 3.3. *Let M be a $\mathbb{C}$-manifold of the form $V \times N$ for a Stein manifold V of positive dimension. Then* $\mathrm{carddim}\,\mathrm{Hol}(M) \geq \mathrm{carddim}\,\mathrm{Hol}(\mathbb{C}) \geq 2^{\aleph_1}$.

Proof. Lemma 3.2 reduces us to the case in which M is a Stein manifold. If $f : M \to \mathbb{C}$ is a nonconstant holomorphic function, then a connected component M' of the preimage of a regular value is a closed submanifold with $\dim_{\mathbb{C}} M' = \dim_{\mathbb{C}} M - 1$. A closed $\mathbb{C}$-submanifold of a Stein manifold is a Stein manifold [GuRo, p. 210], so we may repeat the process, eventually obtaining a closed embedding $\iota : S \hookrightarrow M$ with S a connected, one-dimensional Stein manifold, hence a connected, noncompact Riemann surface. Now if Y is a closed $\mathbb{C}$-submanifold of a Stein manifold X then the map $\operatorname{Hol} X \to \operatorname{Hol} Y$ obtained by restricting holomorphic functions to Y is surjective [GuRo, Thm. VIII.18], so $\iota^* : \operatorname{Hol}(M) \to \operatorname{Hol}(S)$ is surjective. By Remark 1.1b) and Theorem 3.1, we have $\operatorname{carddim} M \geq \operatorname{carddim} S \geq 2^{\aleph_1}$. □

4. Final Remarks

4.1. **A little set theory.**

For a $\mathbb{C}$-manifold M, the ring $\operatorname{Hol}(M)$ is a subring of the ring of all continuous $\mathbb{C}$-valued functions. For any separable topological space X, the set of continuous functions $f : X \to \mathbb{C}$ has cardinality at most $\mathfrak{c}^{\aleph_0} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \times \aleph_0} = 2^{\aleph_0} = \mathfrak{c}$. Since $\mathbb{C} \subset \operatorname{Hol}(M)$, we have $\# \operatorname{Hol}(M) = \mathfrak{c}$. It follows that $\operatorname{Hol}(M)$ has at most $2^{\mathfrak{c}}$ ideals and thus $\operatorname{carddim} \operatorname{Hol}(M) \leq 2^{\mathfrak{c}}$. Moreover

$$\mathfrak{c} = 2^{\aleph_0} \leq 2^{\aleph_1} \leq 2^{\mathfrak{c}}.$$

Whether either inequality is strict is independent of the ZFC axioms, but e.g. the Continuum Hypothesis (CH) gives $\mathfrak{c} < 2^{\aleph_1} = 2^{\mathfrak{c}}$. Thus under CH we have $\operatorname{carddim} \operatorname{Hol}(M) = 2^{\mathfrak{c}}$ for any Stein manifold M. It may well be the case that the determination of $\operatorname{carddim} \operatorname{Hol}(\mathbb{C})$ is independent of the ZFC axioms.

4.2. **A little history.**

Ideal theory in rings of holomorphic functions was initiated by Helmer, who showed that $\operatorname{Hol}(\mathbb{C})$ is a non-Noetherian domain in which every finitely generated ideal is principal [He40]. A paper of Schilling [Sc46] contains the assertion that every nonzero prime ideal of $\operatorname{Hol}(\mathbb{C})$ is maximal. Kaplansky observed that this is false: there are prime ideals $\mathfrak{p}$ which are nonzero and nonmaximal. In [He53], Henriksen shows that for such a prime ideal $\mathfrak{p}$, the quotient $R/\mathfrak{p}$ is a valuation ring: equivalently, the set of ideals of R containing $\mathfrak{p}$ is linearly ordered under inclusion. The unique maximal ideal $\mathfrak{m}_{\mathfrak{p}}$ containing $\mathfrak{p}$ is *free*: for all $z_0 \in \mathbb{C}$ there is $f \in \mathfrak{m}_{\mathfrak{p}}$ such that $f(z_0) \neq 0$. Every other maximal ideal of $\operatorname{Hol}(\mathbb{C})$ is *fixed*: of the form $\mathfrak{m}_{z_0} = \langle z - z_0 \rangle$ for a unique $z_0 \in \mathbb{C}$, and $\mathfrak{m}_{z_0}$ contains no nonzero prime ideals. Henriksen also shows that for every free maximal ideal $\mathfrak{m}$, the set $\operatorname{Spec} \operatorname{Hol}(\mathbb{C})_{\mathfrak{m}}$ of prime ideals contained in $\mathfrak{m}$ is linearly ordered under inclusion. (In [Al63], Alling sharpens this to the statement that $\operatorname{Hol}(\mathbb{C})_{\mathfrak{m}}$ is a valuation ring.) Moreover he shows that $\# \operatorname{Spec} \operatorname{Hol}(\mathbb{C})_{\mathfrak{m}} \geq 2^{\aleph_1}$. Using an extension of the Mittag-Leffler Theorem due to H. Florack, Alling [Al63] extends these results from $\mathbb{C}$ to any noncompact Riemann surface. Later Alling showed [Al79] that for noncompact Riemann surfaces S and T, the spaces $\operatorname{Spec} \operatorname{Hol}(S)$ and $\operatorname{Spec} \operatorname{Hol}(T)$ are homeomorphic.

The above results are all for complex dimension one, in which noncompact manifolds are Stein. The ideal theory of $\operatorname{Hol}(M)$ for a $\mathbb{C}$-manifold M of dimension at least two seems to have received less attention. In 2012 G. Elencwajg asked [MO] whether there is a $\mathbb{C}$-manifold M with $0 < \operatorname{carddim} \operatorname{Hol}(M) < \aleph_0$. This was answered by Kapovich [Ka15]. Kapovich's first proof of Theorem 1.4 was closely modelled on a criterion used by A. Sasane [Sa08, Thm. 2.2] to show (again) that the Krull dimension of the ring of holomorphic functions on a domain $\Omega \subset \mathbb{C}$ infinite.

I believe Sasane's proof is faulty: it seems to assume that Cohen's Multiplicative Avoidance Theorem [K, Thm. 1.] gives a *unique* ideal, which is not true. I corresponded with Kapovich, and he immediately repaired the argument. In fact, he sent me a version which used ultralimits (in a somewhat different way) to show that if $\operatorname{Hol}(M) \supsetneq \mathbb{C}$, then $\operatorname{carddim} \operatorname{Hol}(M) \geq \aleph_0$. I wrote back to suggest proving the stronger bound $\operatorname{carddim} \operatorname{Hol}(M) \geq \mathfrak{c}$, and he soon did so, rephrasing his argument in terms of hyperreals. Our approach to Theorem 1.4 uses some ideas from [He53].

Added in revision: Professor Sasane saw a copy of this note on my webpage and contacted me about it. He confirmed the mistake in his argument and sent me a draft of a corrigendum, which shows that $\operatorname{Spec} \operatorname{Hol}(\Omega)$ contains arbitrarily long finite chains. He points out that he was inspired by work of von Renteln [vR77]. Thus [Ka15] and the present work also have some indebtedness to von Renteln.

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