

ON BASE SIZE SETS

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ABSTRACT. The base size is a well studied invariant of a faithful permutation representation of a finite group G . In [LMS14], Laison, McNicholas and Seaders defined the base size *set* of G as the set of all base sizes of faithful permutation representations of G . Here we extend their definition to any group G and give a general discussion of this invariant. In particular we explain how to compute it in terms of data of G itself...at least in principle. In practice we use our characterization to compute the base size sets of all dihedral groups, extending a result from [LMS14].

INTRODUCTION

This paper is directly motivated by a recent work of Laison, McNicholas and Seaders [LMS14]. They took the standard notion of the base size of a finite permutation group – see e.g. [BC11] and the references therein – and expanded it into an invariant of an abstract finite group by recording the set of base sizes of all faithful permutation representations of G . We extend this definition to arbitrary groups G : in this setting the base size set $\mathcal{B}(G)$ is a set of cardinal numbers. This makes its computation look impractical. Our main result, Theorem 3.1, addresses this by characterizing the base size set in terms of G alone. Thus if for instance G is finite, the computation of $\mathcal{B}(G)$ is in principle reduced to a finite computation. We carry this out in practice by computing $\mathcal{B}(G)$ for all dihedral groups, which partially answers [LMS14, §1, Question 1]. We also compute $\mathcal{B}(G)$ for every finitely generated commutative group, generalizing the computation of $\mathcal{B}(G)$ for finite commutative groups in [LMS14]. Finally we show that certain interesting classes of groups are obtained by imposing conditions on the base size set, e.g. we characterize groups for which $\mathcal{B}(G) = \{1\}$ and groups for which $2 \in \mathcal{B}(G)$.

1. PERMUTATION REPRESENTATIONS

1.1. Basic definitions.

Let X be a set. We denote by $\text{Sym } X$ the set of all bijections $f : X \rightarrow X$, endowed with the structure of a group under composition of functions. A bijection $\iota : X_1 \rightarrow X_2$ induces an isomorphism $\text{Sym } \iota : \text{Sym } X_1 \rightarrow \text{Sym } X_2$, namely

$$\sigma \in \text{Sym } X_1 \mapsto \iota \circ \sigma \circ \iota^{-1} \in \text{Sym } X_2.$$

So if X is finite of cardinality n , then choosing a bijection from X to $\{1, \dots, n\}$ induces an isomorphism from $\text{Sym } X$ to the usual symmetric group S_n .

A **permutation representation** of a group G is a homomorphism $\Phi : G \rightarrow \text{Sym } X$ for some set X . This notation is a bit heavy: in practice, for $x \in X$, we will write gx for $\Phi(g)(x)$. A permutation representation is **faithful** if Φ is injective. Thus a faithful permutation representation embeds G as a subgroup of $\text{Sym } X$.

Let $\Phi : G \rightarrow \text{Sym } X$ be a permutation representation, and let $Y \subset X$. We say that Y is **stabilized** by Φ (resp. **pointwise fixed** by Φ) if for all $g \in G$ and $y \in Y$, we have $gy \in Y$ (resp. $gy = y$). Let

$$\text{Stab}(Y) := \{g \in G \mid gY \subset Y\},$$

$$\text{Fix}(Y) := \{g \in G \mid \forall y \in Y, gy = y\}.$$

For Josh Laison.

Evidently we have $\text{Fix}(Y) \subset \text{Stab}(Y)$. When $Y = \{y\}$ the two sets coincide: we write $\text{Stab}(y)$ and call it the **point stabilizer**. We observe that

$$\text{Fix}(Y) = \bigcap_{y \in Y} \text{Stab}(y)$$

and that Φ is faithful iff $\text{Fix}(X) = \{e\}$. If $\Phi : G \rightarrow \text{Sym } X$ is a permutation representation, $Y \subset X$ and $g \in G$, then we have

$$(1) \quad \text{Fix}(gY) = g \text{Fix}(Y) g^{-1}.$$

In particular, if for $x, y \in X$ and $g \in G$ we have $gx = y$ then we have

$$(2) \quad \text{Stab } y = g(\text{Stab } x)g^{-1}.$$

Two permutation representations $\Phi_1 : G \rightarrow \text{Sym } X_1$ and $\Phi_2 : G \rightarrow \text{Sym } X_2$ are **equivalent** if there is a bijection $\iota : X_1 \rightarrow X_2$ such that

$$\Phi_2 = (\text{Sym } \iota) \circ \Phi_1.$$

1.2. Some permutation representations.

1.2.1. *The Cayley representation.* For a group G , the **(left) Cayley representation** is

$$\Phi_C : G \rightarrow \text{Sym } G, \quad g \mapsto \Phi(g) : h \mapsto gh.$$

It is simply transitive – that is, for all $x, y \in X$, there is a unique $g \in G$ such that $gx = y$. It follows that $\text{Stab}(x) = \{e\}$ for all $x \in X$, so Φ_C is faithful. Conversely, any simply transitive permutation representation $\Phi : G \rightarrow \text{Sym } X$ is equivalent to the Cayley representation: choose $x_0 \in X$, and define $\iota : X \rightarrow G$ by mapping $x \in X$ to the unique $g \in G$ such that $x = gx_0$. Then $(\text{Sym } \iota) \circ \Phi = \Phi_C$.

1.2.2. *Cayley-Schreier representations.* Let H be a subgroup of a group G . Let $G/H = \{gH \mid g \in G\}$ be the set of left cosets of H in G . We define the **(left) Cayley-Schreier representation**

$$\Phi_{CS} : G \rightarrow \text{Sym } G/H, \quad g \mapsto \Phi(g) : xH \mapsto gxH.$$

The Cayley-Schreier representation Φ_{CS} is transitive: for all $x, y \in X$, there is at least one $g \in G$ such that $gx = y$. Conversely, if $\Phi : G \rightarrow \text{Sym } X$ is a transitive representation, then for any $x_0 \in X$, Φ is equivalent to the Cayley-Schreier representation associated to $H = \text{Stab}(x_0)$: namely, for $x \in X$, by transitivity there is $g_0 \in G$ such that $g_0 x_0 = x$, and we have

$$\{g \in G \mid gx_0 = x\} = g_0 \text{Stab}(x_0).$$

This defines a bijection (“**Orbit-Stabilizer Theorem**”)

$$\iota : X \rightarrow G/\text{Stab}(x_0), \quad x \mapsto g_0 \text{Stab}(x_0),$$

and we have

$$(\text{Sym } \iota) \circ \Phi = \Phi_{CS}.$$

(Using right cosets instead one gets the *right* Cayley-Schreier representation, and when $H = \{e\}$ the right Cayley representation. One sees – e.g. by the above characterization – that the right Cayley-Schreier representation is equivalent to the left Cayley-Schreier representation, so it gives nothing new.)

For a subgroup $H \subset G$, (2) gives that the kernel of $\Phi_{CS} : G \rightarrow \text{Sym } G/H$ is

$$\bigcap_{g \in G/H} \text{Stab}(gH) = \bigcap_{g \in G/H} g \text{Stab}(H) g^{-1} = \bigcap_{g \in G/H} g H g^{-1} = \bigcap_{g \in G} g H g^{-1}.$$

For a subgroup H of a group G , we call $\bigcap_{g \in G} g H g^{-1}$ the **normal core** of H and denote it by $\text{Core}(H)$: it is the largest subgroup of H that is normal in G . Thus the Cayley-Schreier representation associated to H is faithful iff H is **corefree**: $\text{Core}(H) = \{e\}$. If H is itself normal, then $H = \text{Core}(H)$, and in this case the Cayley-Schreier representation is only faithful if $H = \{e\}$.

Example 1.1. Consider the standard action of S_n on $\{1, \dots, n\}$ corresponding to the identity map $S_n \rightarrow \text{Sym}\{1, \dots, n\}$. It is transitive and faithful. Let $H = \text{Stab}(n)$; then H consists of all bijections of $\{1, \dots, n-1\}$ so can be identified with S_{n-1} . By faithfulness, S_{n-1} is a corefree subgroup of S_n .

More generally: for any set X , let $G = \text{Sym } X$ and let $\Phi : G \rightarrow \text{Sym } X$ be the identity map. Clearly Φ is faithful and transitive, so it is equivalent to the Cayley-Schreier representation associated to a corefree subgroup H . Choose $x_0 \in X$ and put $H = \text{Stab}(x_0) = \text{Sym}(X \setminus \{x_0\})$. Then indeed we have

$$\bigcap_{g \in G} gHg^{-1} = \bigcap_{g \in G} \text{Stab}(gx_0) = \bigcap_{g \in G} \text{Sym}(X \setminus \{gx_0\}) = \text{Sym } \emptyset = \{e\}.$$

1.2.3. Coproduct of permutation representations. Let $\{\Phi_i : G \rightarrow \text{Sym } X_i\}_{i \in I}$ be an indexed family of permutation representations of the same group G . Let $X = \coprod_{i \in I} X_i$ (disjoint union). Then there is a natural permutation representation $\Phi : G \rightarrow \text{Sym } X$: for all $g \in G$, $\Phi(g)$ stabilizes X_i and acts on it via Φ_i . We call this the **coproduct** (a.k.a. **direct sum**, **disjoint union**) of the representations (G, Φ_i) . It is faithful iff Φ_i is faithful for at least one $i \in I$.

For a permutation representation $\Phi : G \rightarrow \text{Sym } X$, we introduce an equivalence relation on X : $x \sim y$ iff there is $g \in G$ such that $gx = y$. The equivalence class of x is written Gx and called the **orbit of x under G** . Each orbit Gx is stabilized by G and thus gives a permutation representation $\Phi_{Gx} : G \rightarrow \text{Sym } Gx$. Let \mathcal{O} be the set of all G -orbits on X . Thus $X = \coprod_{Gx \in \mathcal{O}} Gx$ and Φ is the coproduct of the representations Φ_{Gx} . Moreover each Φ_{Gx} is transitive, so as above is equivalent to a Cayley-Schreier representation. (Once again, this is the Orbit-Stabilizer Theorem.) Thus we have shown the following structural result for permutation representations.

Theorem 1.2. *Every permutation representation is equivalent to a coproduct of Cayley-Schreier representations.*

2. BASES AND BASE SIZES

2.1. The base size.

Let $\Phi : G \rightarrow \text{Sym } X$ be a permutation representation, and let Y be a subset of X . We say that Y is a **base** for the permutation representation (G, Φ) if $\text{Fix}(Y) = \{e\}$: the only element of G that fixes every element of Y is e . If Y_1 is a base and $Y_2 \supset Y_1$, then Y_2 is also a base. Bases exist iff X is a base iff Φ is faithful. Henceforth we consider only faithful permutation representations.

A base Y is **irredundant** if no proper subset is also a base. A base Y is **minimal** if there is no base Z with $\#Z < \#Y$. If there is a base there is a minimal base, just because any nonempty set of cardinal numbers has a least element. Clearly every finite base contains an irredundant base and a minimal finite base is irredundant.¹ This cannot generally hold for infinite bases: if Y is a proper, infinite base, then adding to Y a finite number of elements of X yields a redundant base of the same cardinality as Y , hence still minimal.

Here is one motivation for studying bases: let $\Phi : G \rightarrow \text{Sym } X$ be a permutation representation, and let $Y \subset X$ be a base. Then, for all $\sigma_1, \sigma_2 \in G$, if $\sigma_1|_Y = \sigma_2|_Y$, then for all $y \in Y$ we have $\sigma_2^{-1}\sigma_1(y) = y$ and thus $\sigma_2 = \sigma_1$. Thus the effect of G on X can be completely understood in terms of where it sends the elements of Y . If Y is moreover G -stable, then the entire permutation representation can be studied in terms of its effect on Y : more precisely, we get a factorization

$$\Phi : G \hookrightarrow \text{Sym } Y \hookrightarrow \text{Sym } X.$$

The **base size** $\mathfrak{b}(G, \Phi)$ of a faithful permutation representation (G, Φ) is the cardinality of a minimal base.

¹Later we will see an example of a permutation representation $\Phi : \mathbb{Z} \rightarrow \text{Sym } X$ with X countably infinite and such that a subset of X is a base iff it is infinite, and thus there are no irredundant bases.

Example 2.1 (Cayley representation of G). *Let G be a nontrivial group, and let $\Phi_G : G \rightarrow \text{Sym } G$ be the Cayley representation. Since every point stabilizer is trivial, every nonempty subset Y of G is a base, and thus $\mathfrak{b}(G, \Phi_G) = 1$.*

Example 2.2 (Standard representation of $\text{Sym } X$). *Let X be a nonempty set, let $G = \text{Sym } X$, and let $\Phi : G \rightarrow \text{Sym } X$ be the identity map. For all $Y \subset X$, we have $\text{Fix}(Y) = \text{Sym}(X \setminus Y)$, so a subset of X is a base iff $\#(X \setminus Y) \leq 1$. Thus the base size is $\#X - 1$.*

Example 2.3 (Standard representation of $\text{GL}(V)$). *Let k be a field and V a k -vector space. Let $\text{GL}(V)$ be the group of invertible k -linear endomorphisms of V . The natural inclusion $\Phi : \text{GL}(V) \rightarrow \text{Sym } V$ is a faithful permutation representation. For a subset Y of V , we have $\text{Fix}(Y) = \text{Fix}(\text{span}(Y))$. A k -subspace $W \subset V$ is a base iff $W = V$. To avoid trivial cases we assume that $W \subsetneq V$ and $\dim V \geq 2$.*

If $\dim V/W \geq 2$ then one can choose a basis \mathfrak{b}' for W , extend it to a basis \mathfrak{b} for V , and there is a nontrivial permutation of \mathfrak{b} pointwise fixing \mathfrak{b}' , and this extends uniquely to an element of $\text{GL}(V)$ pointwise fixing W but not V . If $\dim V/W = 1$, let \mathfrak{b} be a basis for V such that $\mathfrak{b} \setminus \{b_1\}$ is a basis for W , and let $b_2 \in \mathfrak{b} \setminus \{b_1\}$. Then there is a unique $g \in \text{GL}(V)$ such that g fixes every element of $\mathfrak{b} \setminus \{b_1\}$ and $g(b_1) = b_1 + b_2$. Thus $g \in \text{Fix}(W) \setminus \{e\}$.

We deduce: the bases of V are the spanning subsets of V and the irredundant bases are the k -bases of V .² All irredundant bases have cardinality $\dim V$ and are conjugate under $\text{GL}(V)$. When $\dim V$ is finite, a base is irredundant iff it is minimal. When $\dim V$ is infinite, we may start with any k -basis and take the union with any nonempty subset of size at most $\dim V$ to get a redundant, minimal base.

Each example shows: every cardinal number is the base size of a faithful permutation representation.

2.2. Upper bounds on the base size.

Theorem 2.4. *Let $\Phi : G \rightarrow \text{Sym } X$ be a faithful permutation representation. Then:*

$$(3) \quad \mathfrak{b}(G, \Phi) \leq \#G - 1.$$

Proof. Since Φ is faithful, each nonidentity element $g \in G$ moves some element $x_g \in X$. So if $Y = \{x_g \mid g \in G \setminus \{e\}\}$, then $\mathcal{B}(Y) = \{e\}$. Thus $\mathfrak{b}(G, \Phi) \leq \#G - 1$. \square

Example 2.5. *Let $G = (\mathbb{Z}, +)$. Let $\{a_n\}_{n=1}^\infty$ be a strictly increasing sequence of positive integers. Let $X = \coprod_{n=1}^\infty \mathbb{Z}/a_n\mathbb{Z}$ and let $\Phi : G \rightarrow \text{Sym } X$ be the natural permutation representation, i.e., the coproduct of the Cayley-Schreier representations associated to the subgroups $H_n = a_n\mathbb{Z}$ of \mathbb{Z} . For a subset $Y \subset X$, let I be the set of $n \in \mathbb{Z}^+$ such that Y meets $\mathbb{Z}/a_n\mathbb{Z}$. Then*

$$\text{Fix}(Y) = \bigcap_{n \in I} a_n\mathbb{Z}.$$

If $I = \{n_1, \dots, n_r\}$ is finite, then $\text{Fix}(Y) = \text{lcm}(a_1, \dots, a_{n_r})\mathbb{Z} \supsetneq (0)$, so Y is not a base. On the other hand, if Y is infinite, then $\text{Fix}(Y)$ consists of all integers divisible by infinitely many positive integers, so $\text{Fix}(Y) = (0)$ and Y is a base. Thus:

- *We have $\mathfrak{b}(G, \Phi) = \aleph_0 = \#G = \#G - 1$, so we have equality in (3).*
- *Every infinite set admits a proper infinite subset, so there is no irredundant base.*

Actually, the “minus one” in (3) is a bit silly: when G is infinite, then $\#G = \#G - 1$, whereas if G is finite, (3) can be significantly improved. In this case, every minimal base $Y = \{y_1, \dots, y_b\}$ is irredundant. Put

$$H_0 := G; \quad \forall i \in [1, b], \quad H_i := \bigcap_{j=1}^i \text{Stab}(y_j).$$

We get a descending chain of subgroups

$$(4) \quad G = H_0 \supset H_1 \supset \dots \supset H_b = \{e\}$$

²Presumably the terminology “base” is related to this.

In fact we have $H_i \supsetneq H_{i+1}$ for all $i \in [0, b-1]$: equality would imply that $Y \setminus \{y_{i+1}\}$ is also a base, contradicting irredundance. This shows that $b = \#Y$ is bounded above by the **subgroup length** $\ell(G)$: the length of the longest descending chain of subgroups of G .

Now consider a descending chain

$$G = H_0 \supsetneq H_1 \supsetneq \dots \supsetneq H_r = \{e\}$$

of subgroups of a finite group G , and for $0 \leq i \leq r-1$, put $I_i = [H_i : H_{i+1}]$. Since H_{i+1} is proper in H_i , we have $I_i \geq 2$ for all i and thus $\#G \geq 2^r$, and it follows that

$$\ell(G) \leq \log_2(\#G).$$

For a positive integer n , we denote by $\Omega(n)$ the number of prime divisors with multiplicity. Thus if $n = p_1^{a_1} \cdots p_r^{a_r}$ for primes $p_1 < \dots < p_r$ and $a_1, \dots, a_r \in \mathbb{Z}^+$ we have $\Omega(n) = a_1 + \dots + a_r$. For a finite group G we put

$$\omega(G) := \omega(\#G), \quad \Omega(G) := \Omega(\#G).$$

Since $\#G = \prod_{i=1}^r I_i$ and each $I_i > 1$, we must have $r \leq \Omega(\#G)$, with equality iff each I_i is a prime number. Thus we have shown the following result.

Theorem 2.6. *For a faithful permutation representation of a finite group, we have:*

$$(5) \quad \mathfrak{b}(G, \Phi) \leq \ell(G) \leq \Omega(\#G) \leq \lfloor \log_2(\#G) \rfloor.$$

Example 2.7. *Let p_1, \dots, p_r be prime numbers, and let $G = \prod_{i=1}^r \mathbb{Z}/p_i\mathbb{Z}$. By (5), we have*

$$\mathfrak{b}(G, \Phi) \leq \Omega(\#G) = \Omega(p_1 \cdots p_r) = r$$

for any faithful permutation representation of G . We will now construct such a Φ such that $\mathfrak{b}(G, \Phi) = r$. For $1 \leq i \leq r$, let $H_i = \{(x_1, \dots, x_r) \in G \mid x_i = 0\}$, and let Φ be the coproduct of the Cayley-Schreier representations attached to the subgroups H_i , so $X = \coprod_{i=1}^r G/H_i$. Each H_i is normal in G , so the point stabilizer of any element of G/H_i is H_i . We have $\bigcap_{i=1}^r H_i = \{0\}$ and also that if $J \subsetneq [1, r]$, then $\bigcap_{i \in J} H_i \neq \{0\}$. Thus $Y \subset X$ is a base iff it meets each G/H_i , so $\mathfrak{b}(G, \Phi) = r$. If moreover each $p_i = 2$, then all the inequalities in (5) are equalities.

Example 2.8. *Let p be a prime number, $a \in \mathbb{Z}^+$, and $G = \mathbb{Z}/p^a\mathbb{Z}$. In this case (5) gives $\mathfrak{b}(G, \Phi) \leq a$ for any faithful permutation representation Φ of G , but we can do better. A permutation representation of G is equivalent to a coproduct of copies of the natural representation of G on $\mathbb{Z}/p^b\mathbb{Z}$ for some $0 \leq b \leq a$. If $b \leq a-1$ then $p^{a-1} + p^a\mathbb{Z}$ lies in all the point stabilizers. Thus since Φ is faithful it contains a copy of the Cayley representation, so there is $x \in X$ with $\text{Stab}(x) = \{e\}$ and $\mathfrak{b}(G, \Phi) = 1$.*

The argument of the above example can be generalized.

Lemma 2.9. *Let $\Phi : G \rightarrow \text{Sym } X$ be a faithful permutation representation. For all $x \in X$, we have $\mathfrak{b}(G, \Phi) \leq \mathfrak{b}(\text{Stab}(x), \Phi) + 1$.*

Proof. If Y is a base for $\text{Stab}(x)$, then $Y \cup \{x\}$ is a base for G . □

Lemma 2.10. (Prime Power Trick) *Let G be a finite group with an element of prime power order p^a . Then for any faithful permutation representation Φ of G we have $\mathfrak{b}(G, \Phi) \leq \Omega(\#G) - a + 1$.*

Proof. Let $g \in G$ have order p^a . Since Φ is faithful, $\sigma = \Phi(g)$ has order p^a , so there is $x \in X$ such that the orbit of σ on x has size p^a : otherwise $\sigma^{p^{a-1}} = e$. The Orbit-Stabilizer Theorem gives $\# \text{Stab}(x) \mid \frac{\#G}{p^a}$. Using (5) and Lemma 2.9, we get

$$\mathfrak{b}(G, \Phi) \leq \mathfrak{b}(\text{Stab}(x), \Phi) + 1 \leq \Omega(\# \text{Stab}(x)) + 1 \leq \Omega(\#G) - a + 1. \quad \square.$$

3. THE BASE SIZE SET

For cardinals $\kappa_1 \leq \kappa_2$ we denote by $[\kappa_1, \kappa_2]$ the set of all cardinals κ such that $\kappa_1 \leq \kappa \leq \kappa_2$.

For a group G , the **base size set** $\mathcal{B}(G)$ is the set of all base sizes of all faithful permutation representations of G . Because of (3) we have

$$\mathcal{B}(G) \subset \begin{cases} [0, \#G] & \text{if } G \text{ is infinite} \\ [0, \ell(G)] & \text{if } G \text{ is finite} \end{cases}.$$

(So $\mathcal{B}(G)$ really is a set, even though the class of all faithful permutation representations of G is not!)

Remark 1. a) We have $0 \in \mathcal{B}(G) \iff G = \{e\} \iff \mathcal{B}(G) = \{0\}$.

b) For any nontrivial group G , Example 2.1 gives $1 \in \mathcal{B}(G)$.

3.1. A general description of $\mathcal{B}(G)$. A family \mathcal{F} of subgroups of a group G is a **normal family** if it is closed under conjugation: for all $H \in \mathcal{F}$ and all $x \in G$, $xHx^{-1} \in \mathcal{F}$. For a family \mathcal{F} , the normal family **generated** by \mathcal{F} is

$$\overline{\mathcal{F}} := \{gHg^{-1} \mid g \in G, H \in \mathcal{F}\}.$$

A subfamily \mathcal{G} of \mathcal{F} is just a subset of \mathcal{F} . A family \mathcal{F} of subgroups of a group G is **faithful** if $\bigcap_{H \in \mathcal{F}} H = \{e\}$. If \mathcal{F} is a faithful normal family of subgroups of a group G . A (not necessarily normal) subfamily \mathcal{G} of a faithful normal family is a **minimal subfamily** if it is a faithful subfamily of \mathcal{F} and among all faithful subfamilies of \mathcal{F} it has minimal cardinality.

Now we have the following key result.

Theorem 3.1. *Let G be a group.*

a) *A family \mathcal{F} of subgroups of G is the family of point stabilizers of some permutation representation of G iff \mathcal{F} is a normal family. A family \mathcal{F} of subgroups of G is the family of point stabilizers of a faithful permutation representation of G iff \mathcal{F} is normal and faithful.*

b) *The base size set $\mathcal{B}(G)$ is the set of cardinalities of minimal subfamilies of faithful normal families of subgroups of G .*

Proof. a) By (2), $\{\text{Stab}(x) \mid x \in X\}$ is a normal family in G . If \mathcal{F} is a normal family in G , then let

$$X_{\mathcal{F}} = \coprod_{H \in \mathcal{F}} G/H$$

and let $\Phi_{\mathcal{F}} : G \rightarrow \text{Sym } X_{\mathcal{F}}$ be the associated permutation representation. Then the family of point stabilizers of $\Phi_{\mathcal{F}}$ is \mathcal{F} , and \mathcal{F} is faithful iff $\Phi_{\mathcal{F}}$ is.

b) Let $\Phi : G \rightarrow \text{Sym } X$ be a faithful permutation representation of G , and let $\mathcal{F} = \{\text{Stab}(x) \mid x \in X\}$ be the corresponding normal family of point stabilizers. Every base Y for Φ contains a base $Y' \subset Y$ with pairwise distinct point stabilizers, so $\mathfrak{b}(G, \Phi)$ is the minimal cardinality of a subset $\mathcal{G} \subset \mathcal{F}$ such that $\bigcap_{H \in \mathcal{G}} H = \{e\}$. \square

Thus we have explained how to “compute” the base size set of a group purely in terms of the group itself. This is certainly not a deep result. One may wonder whether it is of any actual use – perhaps it is merely a rephrasing of the problem. Our position is that it is a *useful rephrasing* of the problem, and we aim to justify this in the remainder of the paper. In particular we will use Theorem 3.1 to compute the base size sets of all dihedral groups, answering part of [LMS14, § 5, Question 1].

3.2. A containment result.

Theorem 3.2. *Let G be a group and H a subgroup of G . Suppose that for all subgroups J of H , we have*

$$\{hJh^{-1} \mid h \in H\} = \{gJg^{-1} \mid g \in G\}.$$

In particular, this holds if every subgroup of H is normal in G . Then

$$\mathcal{B}(H) \subseteq \mathcal{B}(G).$$

Proof. Every element κ of $\mathcal{B}(H)$ is the size of a minimal subfamily \mathcal{G} of a faithful normal family \mathcal{F} of subgroups of H . By hypothesis, \mathcal{F} is a faithful normal family of subgroups of G of which \mathcal{G} is again a minimal subfamily, so $\kappa \in \mathcal{B}(G)$. \square

3.3. Products.

Here is a notationally cumbersome but useful reformulation of Example 2.7.

Theorem 3.3. (*Product Theorem*) *Let $\kappa \geq 1$ be a cardinal, and for each $i \in \kappa$, let G_i be a nontrivial group. Let $G = \prod_{i \in \kappa} G_i$ be the direct product, and let*

$$\mathfrak{g} = \bigoplus_{i \in \kappa} G_i := \{(g_i) \in G \mid g_i = e \text{ for all but finitely many } i \in \kappa\}$$

be the direct sum.

a) *We have $\mathcal{B}(G) \supseteq [1, \kappa]$.*

b) *We have $\mathcal{B}(\mathfrak{g}) \supseteq [1, \kappa]$.*

Proof. We will work with the direct product G . To work with the direct sum \mathfrak{g} , simply replace instances of \prod by \bigoplus . Let $\alpha \in [1, \kappa]$. For each $i \in \alpha$, put

$$H_i := \prod_{0 \leq j < i} G_j \times \{e\} \times \prod_{i < j \leq \alpha} G_j \times \prod_{\alpha < j < \kappa} \{e\}.$$

Then each H_i is normal in G , so $\mathcal{F} = \{H_i\}_{i \in \alpha}$ is a normal family in G . Moreover \mathcal{F} is faithful but no proper subfamily is faithful, so by Theorem 3.1 we have $\alpha \in \mathcal{B}(G)$. \square

3.4. Base size sets of commutative groups. In this section we will make use of the structure theory of finitely generated commutative groups. We will recall all necessary definitions and results; for proofs, see e.g. [L, §III.7].

Let G be a finite commutative group. Then G is isomorphic to a direct product of cyclic groups of prime power order: there is a positive integer r , prime numbers p_1, \dots, p_r (not necessarily distinct) and positive integers a_1, \dots, a_r such that

$$G \cong \prod_{i=1}^r \mathbb{Z}/p_i^{a_i} \mathbb{Z}.$$

(Although the isomorphism is not unique, the positive integers $r, p_1, \dots, p_r, a_1, \dots, a_r$ are uniquely determined by G .) We call each summand $\mathbb{Z}/p_i^{a_i} \mathbb{Z}$ an elementary divisor and put $\eta(G) = r$, the “number of elementary divisors of G .”

Theorem 3.4 (Laisson-McNicholas-Seaders [LMS14]). *Let G be a finite commutative group. Then $\mathcal{B}(G) = [1, \eta(G)]$.*

Proof. Step 1: The Product Theorem (Theorem 3.3) applies to give $\mathcal{B}(G) \supseteq [1, \eta(G)]$.

Step 2: We show that $\mathcal{B}(G) \subset [1, \eta(G)]$ by induction on $\eta(G)$, the case $\eta(G) = 1$ being Example 2.8. Now let $\Phi : G \rightarrow \text{Sym } X$ be a faithful permutation representation, and let p be a prime dividing $\#G$, and let $G[p]$ be the subgroup of all elements g such that $pg = 0$. (It is a subgroup since G is commutative.) In order for the point stabilizers to intersect to (0) , there must be some $x \in X$ such that $\text{Stab}(x) \not\supseteq G[p]$. In a finite commutative group H , the number of elementary divisors of H that are p -groups is $\log_p \#H[p]$. Thus, since $\text{Stab}(x)$ does not contain all of $G[p]$, we must have

$$\eta(\text{Stab}(x)) \leq \eta(G[p]) - 1 \leq \eta(G) - 1.$$

By induction, we have $\mathfrak{b}(\text{Stab}(x), \Phi) \leq \eta(G) - 1$, and then Lemma 2.9 gives $\mathfrak{b}(G, \Phi) \leq \eta(G)$. \square

Theorem 3.5. *Let G be infinite, finitely generated and commutative, so*

$$G \cong \mathbb{Z}^a \oplus \bigoplus_{i=1}^r \mathbb{Z}/p_i^{a_i}$$

with $a \geq 1$. We have

$$\mathcal{B}(G) = [1, a + r] \cup \{\aleph_0\}.$$

Proof. Step 1: Since G is countably infinite, by (3) we have $\mathcal{B}(G) \subset [1, \aleph_0]$.

Step 2: Theorem 3.3 gives $[1, a + r] \subset \mathcal{B}(G)$, while Example 2.5 and Theorem 3.2 give $\aleph_0 \in \mathcal{B}(G)$.

Step 3: It remains to show that if $k \in \mathbb{Z}^+ \cap \mathcal{B}(G)$, then $k \leq a + r$. We show this by induction on a : the base for our induction will be $a = 0$, and this case is Theorem 3.4. Now let $Y = \{y_1, \dots, y_k\}$ be a finite base for a permutation representation (G, Φ) . Let H be a subgroup of G . There are $b, b_1, \dots, b_r \in \mathbb{Z}$ with $0 \leq b \leq a$, $0 \leq b_i \leq a_i$ for all i such that $H \cong \mathbb{Z}^b \oplus \bigoplus_{i=1}^r \mathbb{Z}/p_i^{b_i}$. Moreover H has finite index in G iff $b = a$. For $i \in [1, k]$, let $H_i = \text{Stab}(y_i)$. It cannot be that each H_i has finite index in G , since then $\{0\} = \bigcap_{i=1}^k H_i$ has finite index in the infinite group G . So for at least one i we have $b_i \leq a - 1$, and then by induction

$$\mathfrak{b}(G, \Phi) \leq \mathfrak{b}(H_i, \Phi) + 1 \leq a + r. \quad \square$$

Theorem 3.6. *Let κ be an infinite cardinal, and let G be a free commutative group of rank κ : that is, $G \cong \bigoplus_{i \in \kappa} \mathbb{Z}$. Then*

$$\mathcal{B}(G) = [1, \kappa].$$

Proof. Theorem 3.3 gives $\mathcal{B}(G) \supseteq [1, \kappa]$. Since $\#G = \kappa$, we have $\mathcal{B}(G) \subseteq [1, \kappa]$ by (3). \square

3.5. Base sizes one and two.

We call a subgroup M of a group G **surtrivial** if: M is nontrivial, there is no nontrivial subgroup N contained in M , and for all nontrivial subgroups H of G we have $H \supset M$. Equivalently, a subgroup is surtrivial if it is generated by an element $\tau \neq e$ such that every nontrivial subgroup of G contains τ . Note that τ necessarily has prime order. A group can have at most one surtrivial subgroup.

Theorem 3.7. *Let G be a group.*

a) *The following are equivalent:*

(i) *We have $\mathcal{B}(G) = \{1\}$.*

(ii) *G admits a surtrivial subgroup.*

b) *The following are equivalent:*

(i) *We have $2 \in \mathcal{B}(G)$.*

(ii) *There are nontrivial subgroups H_1 and H_2 such that $H_1 \cap H_2$ is trivial.*

Proof. a) (i) \implies (ii): We go by contraposition. Let \mathcal{L} be the lattice of all subgroups of G , and let \mathcal{L}' be the partially ordered subset of all nontrivial subgroups of G . A surtrivial subgroup is precisely a bottom element of \mathcal{L}' . So we assume that \mathcal{L}' has no bottom element. Suppose first that every chain in \mathcal{L}' has a lower bound in \mathcal{L}' . Then by Zorn's Lemma \mathcal{L}' has a minimal element H , which is necessarily cyclic of prime order. Being a minimum and not the bottom element means there is a nontrivial subgroup K which does not contain H . Then $H \cap K = \{e\}$. The normal family generated by $\{H, K\}$ corresponds to a faithful representation with base size 2. Next suppose that some chain \mathcal{C} in \mathcal{L}' has no lower bound in \mathcal{L}' . This means that the intersection over the elements of \mathcal{C} is trivial. The normal family generated by \mathcal{C} corresponds to a faithful representation with base size at least 2.

(ii) \implies (i): Since every nontrivial subgroup contains M , the intersection over a family of nontrivial subgroups contains M , so a faithful representation has a trivial point stabilizer and $\mathcal{B}(G) = \{1\}$.

b) Because the conjugation does not change whether a subgroup is trivial, we have $2 \in \mathcal{B}(G)$ iff there is a normal family of nontrivial subgroups some two of which intersect trivially iff there are two nontrivial subgroups that intersect trivially. \square

Corollary 3.8. *For a nontrivial finite group G , the following are equivalent:*

(i) *G is cyclic of prime power order or a generalized quaternion group Q_{2^n} for some $n \geq 3$.*

(ii) *We have $\mathcal{B}(G) = \{1\}$.*

(iii) *We have $2 \notin \mathcal{B}(G)$.*

³See [C1] for an elementary treatment of the generalized quaternion groups.

Proof. (i) \iff (ii): A finite group G admits a surtrivial subgroup iff G is cyclic of prime power order or a generalized quaternion group Q_{2^n} for $n \geq 3$ [R, 5.3.6].

(ii) \implies (iii): This is clear.

(iii) \implies (ii): A nontrivial finite group contains at least one minimal nontrivial subgroup (these are precisely the subgroups of prime order). If there is exactly one such group, it is surtrivial and $\mathcal{B}(G) = \{1\}$; otherwise, the intersection of any two of them is trivial, so $2 \in \mathcal{B}(G)$ by Theorem 3.7b). \square

Example 3.9. Let p be a prime number, and let C_{p^∞} be the **Prüfer p -group**: the group of all roots of unity in \mathbb{C} of order a power of p . This group has for each $i \in \mathbb{Z}^+$ a unique subgroup H_i of size p^i – namely, the (p^i) th roots of unity, and $C_{p^\infty} = \bigcup_i H_i$. In fact, the H_i ’s are the only proper, nontrivial subgroups of C_{p^∞} , so H_1 is a surtrivial subgroup of C_{p^∞} . Thus $\mathcal{B}(C_{p^\infty}) = \{1\}$.

Example 3.10. For all $n \geq 3$ there are injective group homomorphisms $Q_{2^n} \hookrightarrow Q_{2^{n+1}}$, and we can define the **infinite quaternion group** as the direct limit

$$Q_{2^\infty} = \varinjlim Q_{2^n}.$$

This is an infinite 2-group with a unique element τ of order 2, and thus $\langle \tau \rangle$ is a surtrivial subgroup. Thus $\mathcal{B}(Q_{2^\infty}) = \{1\}$.

Let G be an infinite group admitting a surtrivial subgroup $M = \langle \tau \mid \tau^p = 1 \rangle$. Then G is an infinite p -group: if $e \neq g \in G$, since $\tau = g^n$ for some nonzero integer n , we have $g^{pn} = e$, so g has finite order. The order of g must be a power of p : otherwise $\langle g \rangle$ has a subgroup which does not contain τ . If G is moreover commutative, every nontrivial finitely generated subgroup of G is a finite commutative p -group with a unique subgroup of order p , hence cyclic of prime power order. It follows easily that $G \cong C_{p^\infty}$. If $p = 2$, Banach showed [Ba10] that G is isomorphic to C_{2^∞} or to Q_{2^∞} . On the other hand, for each prime $p > 10^{75}$, Olshanskii constructed *uncountably many* nonisomorphic finitely generated p -groups with surtrivial subgroups: “extended Tarski monsters.” Thus the precise classification of finitely generated groups G with $\mathcal{B}(G) = \{1\}$ looks hopeless.

3.6. Dedekind groups.

Theorem 3.11 (Downward Closure). *Let G be a group in which every subgroup is normal.*

- a) *Suppose $n \in \mathbb{Z}^+ \cap \mathcal{B}(G)$. Then $[1, n] \subset \mathcal{B}(G)$.*
- b) *Thus if G is finite, then $\mathcal{B}(G) = [1, n]$ for some $n \in \mathbb{Z}^+$.*

Proof. It is enough to show that if $n \geq 2$ and $n \in \mathcal{B}(G)$, then also $n-1 \in \mathcal{B}(G)$. Since $n \in \mathcal{B}(G)$ there is a family \mathcal{F} and subgroups $H_1, \dots, H_n \in \mathcal{F}$ with trivial intersection and such that no intersection of fewer than n of the H_i ’s is trivial. For $1 \leq i \leq n-2$, put $\underline{H}_i = H_i$, and put $\underline{H}_{n-1} = H_{n-1} \cap H_n$. Clearly $\bigcap_{i=1}^{n-1} \underline{H}_i = \{e\}$. Let \mathcal{F} be the normal family generated by $\{\underline{H}_1, \dots, \underline{H}_{n-1}\}$, so \mathcal{F} is the set of point stabilizers of a faithful representation Φ of G . Clearly $\mathfrak{b}(G, \Phi) \leq n-1$. Since an intersection of k elements of \mathcal{F} is an intersection of either k or $k+1$ elements of \mathcal{F} , no intersection of fewer than $n-1$ elements of \mathcal{F} can be trivial, so $\mathfrak{b}(G, \Phi) = n-1$. \square

A **Dedekind group** is a group in which all subgroups are normal. Of course this includes all commutative groups. The smallest noncommutative Dedekind group is Q_8 . Finite Dedekind groups were classified by Dedekind [De97], and infinite Dedekind groups were classified by Baer [Ba33]. The answer is remarkably simple.

Theorem 3.12. *For a noncommutative group G , the following are equivalent:*

- (i) *G is Dedekind.*
- (ii) *There is a torsion commutative group T with no element of order 4 such that $G \cong Q_8 \oplus T$.*

We will use Theorem 3.12 to compute the base size set of any finite noncommutative Dedekind group.

Theorem 3.13. *Let $G = Q_8 \oplus_{i=1}^n \mathbb{Z}/p_i^{a_i} \mathbb{Z}$ for prime powers $p_1^{a_1}, \dots, p_n^{a_n}$ with no $p_i^{a_i}$ divisible by 4. Then $\mathcal{B}(G) = [1, n+1]$.*

Proof. Step 0: Since G is Dedekind, all its Sylow p -subgroups are normal, and thus G is the direct product of its Sylow p -subgroups. It follows that if $d \mid \#G$ is such that $\gcd(d, \frac{\#G}{d}) = 1$ then there is a unique subgroup of G of order d : namely, the product of all the Sylow p -subgroups for primes $p \mid d$.

Step 1: Theorem 3.3 gives $\mathcal{B}(G) \supset [1, n+1]$. Now let $\Phi : G \rightarrow \text{Sym } X$ be any faithful permutation representation of G . We need to show that $\mathcal{B}(G, \Phi) \leq n+1$.

Step 2: First we assume that G is a 2-group, and thus $G = Q_8 \oplus_{i=1}^n \mathbb{Z}/2\mathbb{Z}$. Then the elements $G[2]$ of order dividing 2 form a subgroup of G . (As mentioned above, this holds for all commutative groups. It does not hold for all noncommutative groups, but it holds in Q_8 and thus also in G .) Let $\#G[2] = 2^{a+1}$, and among the elements of any minimal (hence irredundant, since G is finite) base for Φ we must have at least one x such that $H := \text{Stab}(x) \not\supset G[2]$. Then H is a Dedekind 2-group with $\#H[2] = 2^a$ with $a < n+1$. By induction we have $\mathcal{B}(H) = [1, a] \subset [1, n]$, and then by Lemma 2.9 we have $\mathcal{B}(G) \subset [1, n+1]$, completing the proof in this case.

Step 3: Suppose G is not a 2-group, so $p_n > 2$. Lemma 2.10 gives $x \in X$ such that $[G : \text{Stab}(x)] = p_n^{a_n}$. Because G is nilpotent this implies that $H := \text{Stab}(x) \cong Q_8 \oplus_{i=1}^{n-1} \mathbb{Z}/p_i^{a_i} \mathbb{Z}$. By induction we have $\mathcal{B}(H, \Phi) \leq n$ and thus $\mathcal{B}(G, \Phi) \leq n+1$. \square

3.7. Dihedral groups. For $n \in \mathbb{Z}^+$, we define the dihedral group

$$D_n = \langle a, b \mid a^n = b^2 = e, bab^{-1} = a^{-1} \rangle.$$

The sets $\mathcal{B}(D_n)$ are studied in [LMS14]. The cases $n = 1, 2$ are rather degenerate: we get C_2 and $C_2 \times C_2$, respectively. If $n \geq 3$, then $a \neq a^{-1}$ so the group is not abelian. The cyclic group $C_n = \langle a \rangle$ is a subgroup of D_n ; all elements of the complement have order 2. Thus there is no surtrivial subgroup, so $2 \in \mathcal{B}(D_n)$.

For $n \in \mathbb{Z}^+$, let $\omega(n)$ be the number of distinct prime divisors of n . Laison, McNicholas and Seaders show that $\mathcal{B}(D_n) = [1, \omega(n) + 1]$ in the following cases:

- When $n = 1, 2$. (The group D_n is then commutative, and Theorem 3.4 applies.)
- When $n = p^k$ is a prime power [LMS14, Prop. 5].
- When $n = 2p^k$, p an odd prime power [LMS14, Prop. 6].
- When $n = pq$ is a product of two odd primes [LMS14, Cor. 12].

We aim to show it here in general. For this we will need an explicit description of the subgroup lattice of D_n (see e.g. [C2]). Each subgroup of D_n appears exactly once in the following list:

- I. (Cyclic subgroups) For all $d \mid n$, $C_d := \langle a^{n/d} \rangle$ is cyclic of order d .
- II. (Dihedral subgroups) For all $d \mid n$ and $0 \leq i \leq d-1$, $D(\frac{n}{d}, i) := \langle a^d, a^i b \rangle$ is dihedral of order $\frac{2n}{d}$.

Moreover, each C_d is normal in D_n . If n is odd, then two dihedral subgroups of the same order are conjugate. If n is even, then any dihedral subgroup of order $\frac{2n}{d}$ is conjugate to exactly one of $D(\frac{n}{d}, 0) = \langle a^d, b \rangle$ or $D(\frac{n}{d}, 1) = \langle a^d, ab \rangle$.

Theorem 3.14. *For all $n \in \mathbb{Z}^+$, we have $\mathcal{B}(D_n) = [1, \omega(n) + 1]$.*

Proof. Step 1: Write $n = p_1^{a_1} \cdots p_r^{a_r}$ (so $\omega(n) = r$) with $p_1 < \dots < p_r$. We have an element of order $p_r^{a_r}$ in D_n , so by Lemma 2.10, there is an $x \in X$ with an orbit of size $p_r^{a_r}$ and thus $H_1 = \text{Stab}(x)$ has order $2p_1^{a_1} \cdots p_{r-1}^{a_{r-1}}$ and $\mathcal{B}(D_n, \Phi) \leq \mathcal{B}(H_1, \Phi) + 1$. Since every subgroup of a dihedral group is either cyclic or dihedral, we can continue, getting $H_1 \supset H_2 \supset \dots \supset H_{r-1}$ such that $\#H_{r-1} = 2p_1^{a_1}$ and

$$\mathfrak{b}(D_n, \Phi) \leq \mathfrak{b}(H_{r-1}, \Phi) + r - 1.$$

And now H_{r-1} has an element of order $p_1^{a_1}$ whether $p_1 = 2$ or not, so we can continue to H_r of order 2. Then $\mathcal{B}(H_r, \Phi) = \mathcal{B}(C_2, \Phi) = 1$, so

$$\mathfrak{b}(D_n, \Phi) \leq \mathcal{B}(H_r, \Phi) + r = r + 1.$$

Step 2: Consider C_n as a subgroup of D_n : indeed every subgroup of C_n is normal in D_n . So Theorem 3.7 applies to show that

$$\mathcal{B}(D_n) \supset \mathcal{B}(C_n) = [1, \omega(n)].$$

Step 3: For $1 \leq i \leq r$, put $H_i = D(\frac{n}{p_i^{a_i}}, 0)$, and put $H_{r+1} = C_n$, and $\mathcal{F} = \{H_1, \dots, H_r\}$. Let $\overline{\mathcal{F}}$ be the normal family generated by \mathcal{F} . We have $\bigcap_{i=1}^r H_i = D(1, 0) = \langle b \rangle$, so $\bigcap_{H \in \overline{\mathcal{F}}} H \subset \bigcap_{H \in \mathcal{F}} H = \{e\}$, i.e., $\overline{\mathcal{F}}$ is faithful. It is easy to see that no proper subfamily of \mathcal{F} is faithful, but that is not enough: we need to show that no r elements of $\overline{\mathcal{F}}$ intersect to $\{e\}$. For this, we observe first that for $1 \leq i \leq r$, the intersection of any $j \geq 2$ conjugates of H_i is $C_{\frac{n}{p_i^{a_i}}}$. The next observation is that the intersection of any two dihedral subgroups of D_n with coprime indices is again dihedral (and not cyclic). Indeed, suppose the subgroups are $D(\frac{n}{d_1}, i_1)$ and $D(\frac{n}{d_2}, i_2)$ with $\gcd(d_1, d_2) = 1$, and let $x, y \in \mathbb{Z}$ be such that $xd_1 - yd_2 = i_2 - i_1$. Then

$$a^{i_1 + xd_1} b = a^{i_2 + yd_2} b \in D\left(\frac{n}{d_1}, i_1\right) \cap D\left(\frac{n}{d_2}, i_2\right).$$

It follows that $\overline{\mathcal{F}}$ has no faithful subfamily of size less than $r + 1$: intersecting two dihedral elements of $\overline{\mathcal{F}}$ of the same index has the same effect as intersecting one of the elements with $H_{n+1} = C_n$, so without enlarging the size of a faithful subfamily we may assume that it contains H_{n+1} and does not contain any two conjugate subgroups. Thus it consists of H_{n+1} and a family of dihedral subgroups H_{i_1}, \dots, H_{i_s} with indices d_1, \dots, d_s distinct prime power divisors of n , so $\bigcap_{j=1}^s H_{i_j}$ has index at most $d_1 \cdots d_s \mid n$ in a subgroup of order $2n$ and thus is nontrivial. \square

Theorem 3.15. *For the infinite dihedral group $D_\infty = \langle a, b \mid b^2 = e, bab^1 = a^{-1} \rangle$ we have $B(D_\infty) = \{1, 2, \infty\}$.*

Proof. Let $C_\infty = \langle a \rangle$. Then C_∞ is infinite cyclic and hence index 2 in D_∞ . Every subgroup of C_∞ is normal in D_∞ , so by Theorem 3.2 we have

$$\mathcal{B}(D_\infty) \supset \mathcal{B}(C_\infty) = \{1, \aleph_0\}.$$

Every element of $D_\infty \setminus C_\infty$ has order 2 and there are two conjugacy classes of such subgroups. In particular there is more than one subgroup of order 2, hence by Theorem 3.7 we have $2 \in \mathcal{B}(D_\infty)$. By (3) we have

$$\mathcal{B}(D_\infty) \subset [1, \#D_\infty] = [1, \aleph_0].$$

Thus it remains to show that no $3 \leq n \leq \aleph_0$ lies in $\mathcal{B}(G)$. So let \mathcal{F} be a faithful normal family in D_∞ . As usual, if $\{e\} \in \mathcal{F}$, then the size of a minimal subfamily is 1, so assume not. If \mathcal{F} contains an order 2 subgroup H , then there is $g \in G$ such that $H \neq gHg^{-1} \in \mathcal{F}$, so $H \cap gHg^{-1} = \{e\}$, and the size of a minimal subfamily is 2. So suppose that \mathcal{F} contains a subgroup H of order at least 3. Then $H \cap C_\infty$ is nontrivial, so in fact H has finite index in C_∞ and thus also in D_∞ . Thus \mathcal{F} consists of finite index subgroups, so if \mathcal{F} itself is finite, then $\bigcap_{H \in \mathcal{F}} H$ has finite index in D_∞ hence is nontrivial. Thus $\#\mathcal{F} \geq \aleph_0$. \square

3.8. A lattice-theoretic perspective.

For a Dedekind group G , Theorem 3.1 reduces the computation of $\mathcal{B}(G)$ to a question about the lattice $\text{Sub}(G)$ of subgroups of G . One can formulate the notion of a base size set in an arbitrary complete lattice, as follows: Let (\mathcal{L}, \leq) be a complete lattice, with bottom element 0. A **0-family** is a subset $\mathcal{F} \subset \mathcal{L}$ such that $\inf \mathcal{F} = 0$, i.e., the only $z \in \mathcal{L}$ such that $z \leq x$ for all $x \in \mathcal{F}$ is $z = 0$. A **base** for a 0-family is a subfamily that is also a 0-family. The **base size** of a 0-family \mathcal{F} is the minimal cardinality of a base of \mathcal{F} , and the **base size set** $\mathcal{B}(\mathcal{L})$ is the set of all base sizes of 0-families in \mathcal{L} .

Remark 2. a) In a finite chain, a 0-family contains 0, so the base size set is $\{1\}$.

b) Let $\mathcal{L} = \{0, 1\}^n$ be the hypercube: i.e., Cartesian product of n copies of $\{0, 1\}$, with the product ordering. (This is the lattice of subgroups of $\mathbb{Z}/(p_1 \cdots p_n \mathbb{Z})$, $p_i \neq p_j$). We can then “see the bases,” which can have size between 1 and n .

c) For a nonzero k -vector space V , let $\mathcal{L}(V)$ be the lattice of k -subspaces of V . When $k = \mathbb{Z}/p\mathbb{Z}$, this is the complete lattice associated to a direct sum of copies of $\mathbb{Z}/p\mathbb{Z}$. It is not hard to show that the base size set of $\mathcal{L}(V)$ is $[1, \dim V]$.

4. QUESTIONS

Question 1. *Is it true that for all subgroups $H \subset G$ we have $\mathcal{B}(H) \subset \mathcal{B}(G)$?*

Question 2. *a) Does $\mathcal{B}(G)$ have a maximum for every group G ?*

b) If G is infinite, $\mathcal{B}(G)$ need not be downward closed: e.g. $\mathcal{B}(\mathbb{Z}) = \{1, \aleph_0\}$. Must $\mathcal{B}(G)$ be downward closed if G is finite? Already we do not know whether $4 \in \mathcal{B}(G)$ implies $3 \in \mathcal{B}(G)$. It may be worth searching by computer for a counterexample.

Question 3. *Is it true that if G_1 and G_2 are groups, then*

$$\sup(\mathcal{B}(G_1 \times G_2)) = \sup \mathcal{B}(G_1) + \sup \mathcal{B}(G_2)?$$

Question 4. *What are the base size sets of the symmetric groups? It seems that almost nothing is known. For instance, the standard representation of S_n shows that $n - 1 \in \mathcal{B}(S_n)$ for all $n \in \mathbb{Z}^+$. For $n = 5$, the Petersen graph has automorphism group S_5 and the faithful representation of S_5 on its vertex set has base size 3 [GL09, Prop. 17]. But in general it does not seem clear whether $n - 2 \in \mathcal{B}(S_n)$.*

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