COVERING NUMBERS IN LINEAR ALGEBRA

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ABSTRACT. We compute the minimal cardinality of a covering (resp. an irredundant covering) of a vector space over an arbitrary field by proper linear subspaces. Analogues for affine linear subspaces are also given.

1. LINEAR COVERINGS

Let V be a vector space over a field K. A **linear covering** of V is a collection $\{W_i\}_{i \in I}$ of proper K-subspaces such that $V = \bigcup_{i \in I} W_i$. A linear covering is **irredundant** if for all $J \subsetneq I$, $\bigcup_{i \in J} W_i \neq V$. Linear coverings exist iff $\dim V \geq 2$.

The **linear covering number** $\mathrm{LC}(V)$ of a vector space V of dimension at least 2 is the least cardinality $\#I$ of a linear covering $\{W_i\}_{i \in I}$ of V. The **irredundant linear covering number** $\mathrm{ILC}(V)$ is the least cardinality of an irredundant linear covering of V. Thus $\mathrm{LC}(V) \leq \mathrm{ILC}(V)$.

The main result of this note is a computation of $\mathrm{LC}(V)$ and $\mathrm{ILC}(V)$.

Main Theorem. *Let V be a vector space over a field K, with $\dim V \geq 2$.*
a) If at least one of $\dim V$, $\#K$ is finite, then $\mathrm{LC}(V) = \#K + 1$.
b) If $\dim V$ and $\#K$ are both infinite, then $\mathrm{LC}(V) = \aleph_0$.
c) In all cases we have $\mathrm{ILC}(V) = \#K + 1$.

Here is a counterintuitive consequence: the vector space $\mathbb{R}[t]$ of polynomials has a countably infinite linear covering – indeed, for each $n \in \mathbb{Z}^+$, let W_n be the subspace of polynomials of degree at most n. However any irredundant linear covering of $\mathbb{R}[t]$ has cardinality $\#\mathbb{R}+1 = 2^{\aleph_0}$. Redundant coverings can be much more efficient!

The fact that a finite-dimensional vector space over an infinite field cannot be a finite union of proper linear subspaces is part of the mathematical folkore: the problem and its solution appear many times in the literature. For instance problem 10707 in the American Mathematical Monthly is intermediate between this fact and our main result. The editorial comments given on page 951 of the December 2000 issue of the Monthly give references to variants of this fact dating back to 1959. Like many pieces of folklore, there seems to a be mild stigma against putting it in standard texts; an exception is [2, Thm. 1.2].

There are two essentially different arguments that establish this fact. Upon examination, each of these yields a stronger result, recorded as Theorem 4 and Theorem 5 below. From these two results the Main Theorem follows easily.

The first two parts of the Main Theorem have appeared in the literature before (but only very recently!): they were shown by A. Khare [1]. I found these results independently in the summer of 2008. The computation of the irredundant linear covering number appears to be new. The proof of the Main Theorem is given in §2.

There is an analogous result for coverings of a vector space by affine linear subspaces. This is stated in §3; we then briefly discuss what modifications must be made in the proof of the Main Theorem to obtain this affine analogue.

2. Proof of the Main Theorem

First three simple lemmas, all consequences or special cases of the Main Theorem.

Lemma 1. *(Quotient Principle) Let V and W be vector spaces over a field K with $\dim V \geq \dim W \geq 2$. Then $\mathrm{LC}(V) \leq \mathrm{LC}(W)$ and $\mathrm{ILC}(V) \leq \mathrm{ILC}(W)$.*

Proof. By standard linear algebra, the hypothesis implies that there is a surjective linear map $q : V \to W$. If $\{W_i\}_{i\in I}$ is a linear covering of W, then the complete preimages $\{q^{-1}(W_i)\}_{i\in I}$ give a linear covering of V. The preimage of an irredundant covering is easily seen to be irredundant. $\square$

Lemma 2. *For any field K, the unique linear covering of K^2 is the set of all lines through the origin, of cardinality $\#K + 1$. It is an irredundant covering.*

Proof. The set of lines through the origin is a linear covering of K^2. Conversely, any nonzero $v \in K^2$ lies on a unique line, so all lines are needed. The lines through the origin are $\{y = \alpha x \mid \alpha \in K\}$ and $x = 0$, so there are $\#K + 1$ of them. $\square$

Lemma 3. *Let V be a vector space over a field K, with $\dim V \geq 2$. Then there are at least $\#K + 1$ hyperplanes – i.e., codimension one linear subspaces – in V.*

Proof. When $\dim V = 2$ there are exactly $\#K + 1$ hyperplanes $\{L_i\}$. In general, take a surjective linear map $q : V \to K^2$; then $\{q^{-1}(L_i)\}$ are hyperplanes in V. $\square$

The exact number of hyperplanes in V is of course known, but not needed here.

Theorem 4. *Let V be a finite dimensional vector space over a field K, and let $\{W_i\}_{i\in I}$ be a linear covering of V. Then $\#I \geq \#K + 1$.*

Proof. Since every proper subspace is contained in a hyperplane, it suffices to consider hyperplane coverings. We go by induction on d, the case $d = 2$ being Lemma 2. Assume the result for $(d-1)$-dimensional spaces and, seeking a contradiction, that we have a linear covering $\{W_i\}_{i\in I}$ of K^d with $\#I < \#K + 1$. By Lemma 3, there is a hyperplane W which is not equal to W_i for any $i \in I$. Then $\{W_i \cap W\}_{i\in I}$ is a covering of $W \cong K^{d-1}$ by at most $\#I < \#K + 1$ hyperplanes, contradiction. $\square$

Theorem 5. *Let V be a vector space over a field K, and let $\{W_i\}_{i\in I}$ be an irredundant linear covering of V. Then $\#I \geq \#K + 1$.*

Proof. Let $\{W_i\}_{i\in I}$ be an irredundant linear covering of V. Choose one of the subspaces in the covering, say $W_\bullet$. By irredundancy, there exists $u \in W_\bullet \setminus \bigcup_{i\neq\bullet} W_i$; certainly there exists $v \in V \setminus W_\bullet$. Consider the affine line $\ell = \{tu + v \mid t \in K\}$; evidently $\#\ell = \#K$. If $w = tu + v \in \ell \cap W_\bullet$, then $v = w - tu \in W_\bullet$, contradiction. Further, if for any $i \neq \bullet$ we had $\#(\ell \cap W_i) \geq 2$, then we would have $\ell \subset W_i$ and thus also the K-span of ℓ is contained in W_i, so then $u = (2u + v) - (u + v) \in W_i$, contradiction. It follows that $\#\ell = \#K \leq \#(I \setminus \{\bullet\})$. $\square$

Proof of the Main Theorem:

Theorem 5 is part c) of the Main Theorem. So it suffices to compute $\mathrm{LC}(V)$.
Case 1: Suppose $2 \leq \dim V < \aleph_0$. By Theorem 4 we have $\mathrm{LC}(V) \geq \#K + 1$, whereas by Lemma 1 and Lemma 2 we have $\mathrm{LC}(V) \leq \mathrm{LC}(K^2) = \#K + 1$.
Case 2: Suppose $\dim V \geq \aleph_0$ and K is finite. Then $\mathrm{LC}(V) \leq \mathrm{LC}(K^2) = \#K + 1 < \aleph_0$. Suppose that V had a linear covering $\{W_i\}_{i=1}^n$ with $n < \#K + 1$. Then, since n is finite, we may obtain an irredundant subcovering simply by removing redundant subspaces one at a time, until we get an irredundant covering by $m \leq n < \#K + 1$ subspaces, contradicting Theorem 5.
Case 3: Suppose $\dim V$ and $\#K$ are both infinite. Consider $W = \bigoplus_{i=1}^{\infty} K$, a vector space of dimension $\aleph_0$. For $n \in \mathbb{Z}^+$, put $W_n := \bigoplus_{i=1}^{n} K$. Then $\{W_n\}_{n=1}^{\infty}$ gives a covering of W of cardinality $\aleph_0$. Since $\dim V \geq \dim W$, by Lemma 4 we have $\mathrm{LC}(V) \leq \aleph_0$. Thus it remains to show that V does not admit a finite linear covering. But once again, if V admitted a finite linear covering it would admit a finite irredundant linear covering, contradicting Theorem 5.

3. Affine Covering Numbers

An **affine covering** $\{A_i\}_{i \in I}$ of a vector space V is a covering by translates of proper linear subspaces. An affine covering is irredundant if no proper subset gives a covering. Irredundant affine coverings exist iff $\dim V \geq 1$. The **affine covering number** $\mathrm{AC}(V)$ (resp. the **irredundant affine covering number** $\mathrm{IAC}(V)$) is the least cardinality of an affine covering (resp. an irredundant affine covering).

Theorem 6. *Let V be a vector space over a field K, with $\dim V \geq 1$.*
a) If $\min(\dim V, \#K)$ is finite, then $\mathrm{AC}(V) = \#K$.
b) If $\dim V$ and $\#K$ are both infinite, then $\mathrm{AC}(V) = \aleph_0$.
c) We have $\mathrm{IAC}(V) = \#K$.

The proof of the Main Theorem goes through with minor modifications. Lemma 1 holds verbatim. The following self-evident result is the analogue of Lemma 2.

Lemma 7. *For any field K, the unique affine covering of K^1 is the set of all points of K, of cardinality $\#K$. It is an irredundant covering.*

Combining these two results we get the analogue of Lemma 3, in which $\#K + 1$ is replaced by $\#K$. To prove the analogue of Theorem 4, note that for two two codimension one affine subspaces W_1, W_2 of a vector space V, then $W_1 \cap W_2$ is either empty or is a codimension one affine subspace in each W_i. In the proof of the analogue of Theorem 5 we use the line $\ell = \{(1-t)u + tv \mid t \in K\}$.

Acknowledgments. This work was partially supported by National Science Foundation grant DMS-0701771.

References

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